

# Scaffolding Instruction for English Language Learners: A Resource Guide for Mathematics

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# Introduction

## Overview

The Common Core State Standards (CCSS) in mathematics establish rigorous expectations for all learners, including English language learners (ELLs). Although these standards present challenges, they create opportunities to more fully incorporate ELLs into standards-based reform.

The CCSS in mathematics include a focus on the mathematical content required for students at each grade level and also include Standards for Mathematical Practice that apply in different ways across all grade levels. The eight Standards for Mathematical Practice are the following:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

To help ELLs master these standards, it will be very important to have materials and methods that more fully support them in acquiring grade-level knowledge and skills. Effective methods for enabling ELLs to meet the CCSS build on approaches that are effective for all students, but they also provide additional support for ELLs who are learning content in an additional language. These sample, scaffolded lessons (prototypes) can be used by teachers as guides to modify other lessons. Coaches and others providing professional development for teachers may also use the prototypes as exemplars. They may be also used to engage teachers in comparing and contrasting the scaffolds in two or more lessons, and making decisions about which scaffolds may be most beneficial to the students in specific classes.

What follows is an overview of flexible, high-yield scaffolds to support ELLs in meeting the standards, followed by a series of four sample lessons from the New York State Education Department curriculum that have been annotated with scaffolding suggestions throughout. These sample (prototype) lessons are as shown in the following table:

Name of Prototype Lesson	Grade	Module	Lesson
Make Series of Longer Than And Shorter Than Comparisons	Kindergarten	3	3
Use Visual Models to Add and Subtract Two Fractions with the Same Units	4	5	16
Proofs of Laws of Exponents	8	3	6
The Power of Exponential Growth	Algebra I	3	5

## General Approach

Some scaffolds are primarily useful for supporting student’s *receptive skills* including their ability to process new concepts, organize ideas, and acquire academic language including new linguistic structures. Other scaffolds are primarily useful in supporting student’s *productive skills* that include communicating their mathematical thinking as well as seeking clarification about math content or language associated with math. These scaffolds may be used with ELLs at any level of English language proficiency, with variations in the levels of support predicated on students’ prior math knowledge and levels of English proficiency. We suggest collaboration between mathematics teachers and bilingual teachers or teachers of English for speakers of other languages (ESOL) who can support mathematics teachers in scaffolding math curriculum and instruction for English language learners.

## Use Scaffolding Techniques and Routines Consistent With the Common Core State Standards and Recent Research

AIR has ensured that the scaffolding techniques and routines are consistent with the New York State P–12 Common Core Learning Standards (NYS CCLS) by aligning them with criteria in the Evaluating Quality Instructional Programs (EQuIP) rubric. EQuIP is a rubric that 35 states are using. The rubric provides criteria to determine the quality and alignment of curriculum to the Common Core. Consistent with the criteria set by EQuIP, the prototyped lessons that follow are aligned to the depth of the Common Core, address key shifts in the Common Core, are responsive to ELL learning needs, and regularly assess whether students are developing standards-based skills.

The scaffolding techniques and routines in these lessons also are consistent with findings from research reported in the recently released Institute of Education Sciences Practice Guide focused on teaching academic content and literacy to English language learners (Baker et al., 2014), as well as from research related to reading for multiple purposes (August & Shanahan, 2006) and the use of home language instruction for helping ELLs develop literacy and content knowledge in English (e.g., Francis, Lesaux, & August, 2006). The scaffolding techniques specific to mathematics also are in alignment with principles set forth by Moschkovich (2014) and Civil (2007). The research-based scaffolding techniques include teaching academic vocabulary intensively across several days using a variety of techniques: integrating oral and written English language instruction into content area teaching; providing regular opportunities to develop

written language skills; building background knowledge; clarifying content delivered in a second language; and capitalizing on students' home language skills and knowledge.

### Teach Academic Vocabulary

In the lessons that follow, vocabulary is selected for instruction because it is important for understanding the content and appears frequently across content at the target grade level. The scaffolding techniques used to teach academic vocabulary in these lessons are consistent with recent research. (Carlo et al., 2004; Lesaux, Kieffer, Faller, & Kelley, 2010; Lesaux, Kieffer, Kelley, & Harris, in press; Silverman & Hines, 2009; Vaughn et al., 2009). The techniques include “using engaging informational texts as a platform for intensive vocabulary instruction; choosing a small set of academic vocabulary words for in-depth instruction; teaching vocabulary in depth using multiple modalities (writing, listening, and speaking); and teaching students word learning strategies to help them independently figure out the meanings of words” (Baker et al., 2014, p. 6).

Introducing new vocabulary should be explicit and intentional and should use definitions written in student-friendly language. Students should be provided with structures to practice their new vocabulary with peers and adults.

Vocabulary should be explained and taught with several key instructional approaches in mind:

- The teacher should provide an accurate auditory imprint of the vocabulary term by pronouncing it clearly several times and having the students repeat it chorally and then to an assigned partner.
- It should be noted when vocabulary words are the same as words in other contexts, or have homophones with different meanings (*some* and *sum*, for example).
- The teacher should identify the part of speech for the vocabulary word and be sure to identify and explain whether the word could be more than one part of speech. The word *square*, for example, can function as a noun (a perfect square), a verb (to square a number), and an adjective (a square tile).
- The teacher should provide the definition in student-friendly language and have students record it in some way, perhaps in a graphic organizer or in an individual glossary.
- Students should be provided with an immediate opportunity to use the word in context, either by speaking to a partner or by writing a sentence or two using the new term.
- Students should be given opportunities to review vocabulary they have been exposed to but may not have committed to memory. For example, at the beginning of a lesson, students could review vocabulary with partners using flash cards or the folded graphic organizer they have created. They could review vocabulary by lesson or in other ways, such as by grammatical form (nouns, verbs, phrases). The cards could have the vocabulary word and perhaps an illustration on the front, and the back could contain a definition, a first-language translation, an exemplary sentence, and questions that engage the students in discussion about the words.
- Students should have structured opportunities to use this key academic vocabulary—both new terms and previously learned terms—across all four modalities (speaking, reading,

writing, and listening) each day. Students also need opportunities to practice pronouncing key terms.

Word banks and word walls are easily visible resources that can provide appropriate and relevant vocabulary to use in speaking or writing about the content. Word walls or banks should be organized in conceptual ways, perhaps by unit, or perhaps by part of speech. Random organizations may be aesthetically pleasing but may not provide the structure ELLs need to reinforce key ideas about the ways the terms work together. To best orient students toward using these resources, teachers should model indicating when they have used or will use terms from the word walls or word banks.

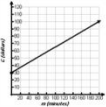
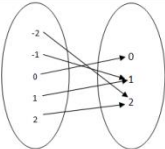
### **Integrate Oral and Written Language Instruction into Content Area Teaching**

The scaffolding techniques used to integrate oral and written language into content area instruction in the lessons that follow are consistent with recent research (August, Branum-Martin, Cardenas-Hagan, & Francis, 2009; Brown, Ryoo, & Rodriguez, 2010; Ryoo, 2009; Silverman & Hines, 2009; Vaughn et al., 2009). Techniques include “strategically using instructional tools such as short videos, visuals, and graphic organizers—to anchor instruction and help students make sense of content; explicitly teaching the content-specific academic vocabulary, as well as the general academic vocabulary that supports it, during content-area instruction; providing daily opportunities for students to talk about content in pairs and small groups; and providing writing opportunities to extend student learning and understanding of the content material” (Baker et al., 2014, p. 6). Other scaffolding techniques are the use of supplementary questions that guide students to the answers for more overarching text-dependent questions, and using structured approaches to Socratic discussions (Thompson & Radosavljevic, 2013).

**Concrete and Visual Models.** For students at the entering, emerging, and transitioning levels of English proficiency, concrete and visual models can make mathematical concepts more apparent and accessible. These models may include manipulatives, illustrations, or other opportunities to have hands-on experiences with the concepts (Fuson, Kalchman, & Bransford, 2005).

**Graphic Organizers and Foldables.** Graphic organizers are visual and graphic displays that present relationships between facts, terms, or ideas within a learning task. Types of graphic organizers include knowledge maps, concept maps, story maps, cognitive organizers, advance organizers, and concept diagrams. Whether they are provided to the students partially completed or students construct and populate them completely themselves depends on the capabilities of the student. Foldables are a type of graphic organizer in which paper is folded in particular ways to layer and selectively reveal information, as in a lift-the-flap book. They are particularly useful for studying new concepts. Graphic organizers support ELLs because they provide a means of displaying complex text succinctly and graphically. Here are a few examples:



Verbal Representation	Graphic Representation	Numeric Representation	Algebraic Representation														
<p>Two plans are comparing cell phone plans. One plan costs \$30 monthly plus \$0.35 per minute.</p>		<table border="1"> <thead> <tr> <th>m</th> <th>C</th> </tr> </thead> <tbody> <tr><td>0</td><td>\$30.00</td></tr> <tr><td>10</td><td>\$33.50</td></tr> <tr><td>20</td><td>\$37.00</td></tr> <tr><td>30</td><td>\$40.50</td></tr> <tr><td>40</td><td>\$44.00</td></tr> <tr><td>50</td><td>\$47.50</td></tr> </tbody> </table>	m	C	0	\$30.00	10	\$33.50	20	\$37.00	30	\$40.50	40	\$44.00	50	\$47.50	$C = 30 + 0.35m$
m	C																
0	\$30.00																
10	\$33.50																
20	\$37.00																
30	\$40.50																
40	\$44.00																
50	\$47.50																
																	

**Scientific Notation**  
Example:

**Standard Notation**  
Example:

How Alike?

---

**Scientific Notation**  
Example:

**Standard Notation**  
Example:

How Different?

---

**CONCLUSION**

**Surface Area**

- measures outside surface
- includes all faces
- is measured in square units
- can be measured using a net

**Volume**

- measures space inside
- includes only space needed to fill the inside
- is measured in cubic units

a way to measure space figures

**Multimedia to Enhance Comprehension.** Although most work in mathematics is text-based, multimedia and visuals can be used as a way to underscore, emphasize, or explain major concepts or fine points. Multimedia should not replace instruction, but the use of media and visuals can help support ELLs’ understanding of a topic that may be unfamiliar to them. Teachers could use short snippets or still shots from carefully selected videos to support ELLs’ understanding of processes. Subtitles (either in English or in students’ first language) with a video clip will provide ELLs with more support.

**Structured Opportunities to Speak With a Partner or Small Group.** When using partner or small-group structures, ELLs should be paired with more proficient English speakers. Also, there should be some initial training to assist pairs in working together. There are several excellent resources that provide guidance related to academic conversations and activities to help students acquire the skills they need to engage in these conversations (Zwiers, 2008; Zwiers & Crawford, 2011). Examples of skills: elaborate and justify, support ideas with examples, build on or challenge a partner’s ideas, paraphrase, and synthesize conversation points.

To assist ELLs, provide students with frames for prompting the skill. For example, for the skill “elaborate and clarify,” frames for prompting the skill might be as follows:

I am a little confused about the part...

Can you tell me more about...?

What do you mean by...?

Can you give me an example?

Frames for responding might be

In other words...

It's similar to when....

Could you try to explain it again?

### **Provide Regular, Structured Opportunities to Write**

The scaffolding techniques used to write in the following lessons are consistent with recent research (Kim et al., 2011; Lesaux et al., in press). Techniques include “providing writing assignments that are anchored in content and focused on developing academic language as well as writing skills; providing language-based supports to facilitate student’s entry into and continued development of writing; using small groups or pairs to provide opportunities for students to work and talk together on varied aspects of writing; and assessing student’s writing periodically to identify instructional needs and provide positive constructive feedback in response”(Baker et al., p. 6). For example, in the prototyped lessons, all writing is anchored in content students have read and focuses on developing academic language through ongoing questions that require students to talk in pairs and then write. All writing assignments provide language-based supports such as graphic organizers to facilitate students’ entry into and continual development of writing and sentence frames and starters.

**Sentence Frames.** Like sentence starters (following), sentence frames have language already provided and a small range of options to complete the sentences (Kinsella, 2012). They may be accompanied by a word bank of relevant vocabulary that students have learned or are practicing. It might be thought of almost as a kind of pull-down menu for each blank, with a group of preselected terms that might work for the specific blank. For example:

This rectangle is divided into \_\_\_\_\_ because it has \_\_\_ parts that are all the same size.

Word bank: halves, thirds, fourths, fifths, sixths, eighths, two, three, four, five, six, eight

**Sentence Starters.** More open-ended than sentence frames, sentence starters can be very supportive and provide structures for students to use in communicating their thinking in a more academic register.

My idea is similar to \_\_\_\_\_’s because \_\_\_\_\_.

First, I \_\_\_\_\_. Next, I \_\_\_\_\_. Finally, I \_\_\_\_\_.

I know the area of parallelogram *B* is smaller than/larger than the area of parallelogram *A* because \_\_\_\_\_.

### **Build Background Knowledge**

**Background Knowledge for Students.** Because ELLs are unlikely to possess a level of background knowledge on par with fluent English speakers, they will need additional support to comprehend the stimulus or prompt texts used in many lessons. Not only might students be unfamiliar with vocabulary words, but they also may be unfamiliar with linguistic constructs and the “real life” contexts used in mathematics problems (that do not relate to the lived experiences of the students). For that reason, there may be times when students will need a succinct and

purposeful explanation of a scenario, illustration, or other information that is not part of the mathematical content or standard related to the lesson. To support students in quickly accessing the background information necessary to understand contexts, the use of short videos and photographs can also be useful.

Teachers should not rely on affirmative student responses to questions about contexts like “You know what a Ferris wheel is, right?” or “Have you ever heard of OPEC?” Because students may be self-conscious about the range and depth of information that is unfamiliar, they may not be entirely candid in publicly identifying information that is not known. Also, students may think they recognize a context but may not fully understand it. For example, when presented with a problem context about a state fair, students may say they knew what it means, “Like when things are equal, they are fair, so it’s one of the United States where things are fair.”

**Background Knowledge for Teachers.** In addition to the scaffolds, teachers themselves will benefit from building their own background knowledge, particularly about the cultures of students and their families and the ways in which mathematics concepts are approached in the communities of origin of the students. Because most teachers were educated in the United States, there may be limited familiarity with educational systems or the organization of learning in systems outside the United States. Although mathematics education has many similar characteristics around the world, there are, at times, significant differences in the ways and order in which mathematics content is presented. Thus, using a strengths-focused approach that highlights what students already know and can do, teachers may learn more about the educational systems in the communities of origin of their students and shift instruction appropriately.

### **Clarify Content Delivered in a Second Language**

As described above it is important to ensure that ELLs understand content presented in their second language. This can be accomplished by clarifying key concepts; through modeling and teacher explanation; and cueing.

**Clarifying Key Concepts.** At times, the linguistic complexity of the language commonly used in mathematics may impede student access to the content being taught. By attending to the language load in the texts shared with students and in the explanations provided orally, key content can be highlighted and clarified (Abedi & Lord, 2001). Some ways to clarify key concepts are rewording text using present tense, shorter sentences, and fewer clauses; using examples related to school contexts; using graphics and arrows to illustrate points; and using white space and color to accentuate important information. Here is an example from the eighth-grade Engage<sup>NY</sup> curriculum, with a scaffolded example for comparison:

#### **Original text**

Suppose a colony of bacteria doubles in size every 8 hours for a few days under tight laboratory conditions. If the initial size is  $B$ , what is the size of the colony after 2 days?

#### **Scaffolded text**

A group of objects doubles every 8 hours.

Today there are  $B$  objects in the group.

How many objects are in the group after 2 days?

(There are 24 hours in one day.)

**Teacher Modeling and Explanation.** Teacher explanation and modeling of thought processes, of the manner in which lesson activities should be carried out, and of high-quality responses will be particularly beneficial for ELLs because explanation and examples enhance comprehension. Explanation and modeling should be used to support students before they are struggling, with teachers clearly explaining each task and modeling an expected student response. Modeling all mathematical discourse that students will be using in their own collaborations with peers and in writing will be beneficial for ELLs at all levels.

**Cueing.** At the beginning of the lesson, include a clear focus on stating the standards, objectives, and agenda for the day, communicated in student-friendly language. This provides an advance schema for the students and allows them to begin to anticipate how the new information will connect to previous learning. In addition, cueing provides reinforcement of key vocabulary. Cueing is recommended for use at the beginning of lessons and again at the end (Wiliam, 2011).

### Capitalize on Student’s Home Language Skills and Knowledge

The scaffolds in the previous sections may be helpful to all students. Scaffolds unique to ELLs include those that capitalize on their home language knowledge and skills to help them acquire knowledge and skills in a new language. A large body of research indicates that ELLs draw on conceptual knowledge and skills acquired in their home language in learning their new language (Dressler, 2006) and that instructional methods that help ELLs draw on home language knowledge and skills promote literacy development in a new language (August et al., 2009; Carlo et al., 2004; Liang, Peterson, & Graves, 2005). In the prototyped lessons that follow, scaffolds that help students draw on home language knowledge and skills include glossaries and side-by-side texts that include home language translations and routines that pair ELLs who are at emerging levels of language proficiency with bilingual partners so that discussions can occur in students’ home language and in English.

**Side-by-Side Texts.** For students who are literate in their first language, a side-by-side text can be useful in helping them comprehend English key vocabulary and language structures. In this example, it is easy for a new speaker of Finnish to isolate the term that represents the English word *hexagon* through the comparison of side-by-side texts.

English	Finnish
Draw a hexagon.	Piirrä kuusikulmio.
This is a regular hexagon.	Tämä on säännöllinen kuusikulmio.
Show your hexagon to your neighbor.	Näyttää kuusikulmio naapurin.

### Differentiate Instruction for Students at Diverse Levels of English Proficiency

The New York State New Language Arts Progressions specify four levels of proficiency and literacy for ELLs—entering, emerging, transitioning, and expanding—and one level of proficiency for ELLs who have just become proficient in English—commanding. In the prototypes that follow, we describe and demonstrate scaffolding appropriate for ELLs at all four

levels of proficiency. ELLs at all levels of proficiency have access to scaffolds that provide multiple means of representation, action and expression, and engagement. Because teachers generally have more than one level of ELL in a group and because within a level of proficiency there are individual differences in knowledge and skills, teachers should reduce the scaffolding to meet the unique needs of individual students.

## Conventions Used to Describe AIR Scaffolding

The purpose of the prototypes is to provide illustrative examples of new activities and additional supports to the original lessons that are beneficial to ELLs. Each prototype includes scaffolds described in the previous section, but with specific connections to the content and intended goals of the lessons. This may include references to specific resources, explanation of specific activities, and scripting of specific teacher language. *AIR New Activity* refers to an activity not in the original lesson that AIR has inserted into the original lesson. For example, Cueing is a new activity AIR has added to the first prototype, the kindergarten lesson titled “Make Series of Longer Than and Shorter Than Comparisons.” *AIR Additional Supports* refer to additional supports added to a component already in place in the original lesson. *AIR new activities* and *AIR additional supports* are boxed whereas the text that is in the original lesson is generally not boxed. *AIR Routines for Teachers* are activities that include instructional conversations that take place between teachers and students. In the *AIR Routines for Teachers* the text of the original lessons appears in standard black, whereas the AIR additions or supports to the lessons are in green.

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# Common Core Inc.



## Kindergarten, Module 3, Lesson 3: Make Series of *Longer Than* and *Shorter Than* Comparisons

### Overview

The following table outlines the scaffolds that have been added to support ELLs throughout the Common Core Inc. Lesson 3: Make a Series of Longer Than and Shorter Than Comparisons.

*AIR New Activity* refers to an activity not in the original lesson that AIR has inserted into the original lesson. *AIR Additional Supports* refer to additional supports added to a component already in place in the original lesson. *AIR new activities* and *AIR additional supports* are boxed whereas the text that is in the original lesson is generally not boxed. *AIR Routines for Teachers* are activities that include instructional conversations that take place between teachers and students. In the *AIR Routines for Teachers* the text of the original Common Core Inc. lessons appears in standard black, whereas the AIR additions to the lessons are in green.

Original Component by Common Core Inc.	AIR Additional Supports	AIR New Activities
<i>None included</i>		Cueing
Say Ten Push-Ups	Teacher background knowledge Key academic vocabulary	
Hidden Numbers	Teacher modeling and explanation	
Make It Equal	<i>None suggested</i>	
Application problem	Background knowledge for students Key academic terms Concrete and visual models Structured opportunities to speak with a partner or smaller group Sentence starters	
Concept development	<i>None suggested</i>	
Problem set	Teacher background knowledge Scaffolded language	
Student debrief	<i>None suggested</i>	
Homework	Homework scaffolds	

### AIR Lesson Introduction

<b>AIR New Activity</b>
<b>Cueing</b> The lesson opens with cueing to provide an anticipatory overview for students (in the form of an

objective and agenda in student-friendly language) and to provide an initial introduction to the key vocabulary of the lesson. This provides an advance schema for the students and allows them to begin to anticipate how the new information will connect to previous learning. The purpose of this cueing is to establish what the lesson will be about and to help students know where to focus their attention throughout the lesson. Because this section is not included in the original lesson, subsequent times allotted for activities have been modified.

### **AIR Routine for Teachers**

Write on board and read aloud: *I will compare the length of different objects. I will use the words “longer than” and “shorter than” to explain what I see.*

T: A few days ago, we compared the heights of different objects. We used the words “taller than” and “shorter than.” Yesterday, we compared length measurements with string. Remember that length is related to how long something is. Length, long. Say it with me: length, long. I can ask, “What is the length?” Or I could ask, “How long?” These mean the same thing. Length, long.

When we compared two things, we noticed what was the same and what was different. Today, we will compare different things. Let’s all say the word compare. (Compare.) Now say it to your neighbor: compare. (Compare.) Today you will compare different objects and use the words longer than and shorter than.

### **Common Core Inc. Say Ten Push-Ups K.NBT.1**

Conduct activity as outlined in GK–M3–Lesson 1(a previous lesson), but now continue to ten 5, encouraging students to predict what comes next in the pattern. This activity extends students’ understanding of numbers to 10 in anticipation of working with teen numbers.

### **AIR Additional Supports**

#### **Background knowledge for teachers**

Teacher background knowledge is provided here to give teachers insight into prior knowledge some students may have. This may help teachers build on the information students bring and attend to opportunities for students to deepen their learning. Note that the time allotted for this lesson is reduced from the original lesson (as noted on the opening page of the lesson) to accommodate time for the cueing activity.

Some languages other than English (such as Japanese and Korean) use a counting system for the teen numbers that names the tens before the ones, just as modeled in this activity. Teachers may help students already able to count into the teens in these languages to make connections between this Push-Up activity and what the students may already know about place value.

**Key academic vocabulary**

Introducing new vocabulary should be explicit and students should be provided with structures to practice their new vocabulary with peers and adults. ELLs should be given opportunities to review vocabulary to which they have been exposed but may not have committed to memory. In addition, ELLs should have structured opportunities to use this key academic vocabulary—both new terms and previously learned terms—across all four modalities (speaking, reading, writing, and listening) each day. Students also need opportunities to practice pronouncing key terms.

To reinforce the words for numbers in English, follow each “push-up” with chorally saying the number twice. “*Ten one! Eleven! Eleven!*” “*Ten five! Fifteen! Fifteen!*”

**Common Core Inc. Hidden Numbers (5 as the Whole) K.OA.3****AIR Additional Supports****Teacher modeling and explanation**

Teacher explanation and modeling of specific steps in a process or thought processes, of the manner in which lesson activities should be carried out, and of high-quality responses will be particularly beneficial for ELLs because explanation and examples enhance comprehension. Explanation and modeling should be used to support students before they are struggling, with teachers clearly explaining each task and modeling an expected student response.

This portion of the lesson includes teacher modeling and explanation as a way to provide comprehensible input for all learners. This modeling will reinforce verbs (like *touch*), culturally specific body motions (like shrugging with palms up to indicate a question), and counting in English. Note that the time allocated for this portion of the lesson has been reduced by 1 minute to accommodate the cueing at the beginning of the lesson. The text of the original lesson is in black, and the new AIR additional supports are in green.

**AIR Routine for Teachers**

Materials: (S) Hidden Numbers mat (template) inserted into personal white boards (See Activity Sheet 4.)

- T: Touch (teacher demonstrates this) and count the fish on your mat. Raise your hand when you know how many (wait for all hands to go up, and then give the signal). *How many fish are there? (Teacher shrugs and holds hands out with palms up, pantomiming a question.)* Ready?
- S: 10.
- T: Put X's on 5 of the fish. We're not going to count those fish right now. Pretend they swam away! (Teacher models by crossing out five fish and counting aloud when crossing them out.) *I cross out one, two, three, four, five. I crossed out five fish. Now you cross out five fish. Go!*
- S: (Cross out 5 fish.)
- T: Circle (teacher draws a circle in the air) a group of 4 from the fish who didn't swim away (teacher points to them on teacher example).
- T: How many fish are left? (Teacher pantomimes a question with a shrug and palms up.)
- S: 1.
- T: Let's circle (teacher draws a circle in the air) that 1. How many did you circle all together?
- S: 5.
- T: Erase your boards. Put X's (teacher pantomimes or models drawing X's) on 5 of the fish again

to show they swam away. How many fish did not (teacher shakes head left to right, indicating “no”) swim away?

S: 5.

T: Now this time circle (teacher draws circle in the air) a group of 2. Circle another 2.

S: (Circle two groups of 2.)

T: How many (teacher pantomimes question) fish have you circled so far?

T: Circle 1 more. Now how many are circled?

T: Erase your boards. Put X’s (pantomime or model drawing X’s) on 5 of the fish again. How many fish can we see?

T: This time circle a group of 3. (Teacher holds up 3 fingers.)

T: Circle a group of 2. (Teacher holds up 2 fingers.)

T: How many (teacher pantomimes question) are in the larger group?

S: 3.

T: How many (teacher pantomimes question) are in the smaller group?

S: 2.

T: How many (teacher pantomimes question) did you circle all together?

S: 5.

Continue this procedure looking for hidden numbers within a group of 6. Pause occasionally to allow students to explain efficient ways of locating the groups.

*Note:* Finding embedded numbers anticipates the work of GK–Module 4 by developing part–whole thinking.

## Common Core Inc. Make It Equal K.CC

Conduct activity as outlined in GK–M3, Lesson 2, but now have students line up their beans (up to 10 beans) in horizontal rows or vertical columns.

*Note:* Students experience comparison visually, a skill crucial to the work of this module.

## Common Core Inc. Application Problem

Draw a monkey with a very long tail. Draw a monkey with a very short tail. Now, draw a yummy banana for the monkeys to share. Is the banana longer than or shorter than the tail of the first monkey? Is it longer than or shorter than the tail of the second monkey? Tell your partner what you notice.

*Note:* The comparison of two different lengths with a neutral object introduces today’s lesson objective.



### Notes on Multiple Means of Representation:

Your below grade level students will benefit from extra practice in determining what objects are longer than and shorter than in order to be ready for comparing two different lengths with a third object in this lesson. You can use interactive technology such as that found at [http://www.kidport.com/GradeK/math/MeasureGeo/MathK\\_Tall.htm](http://www.kidport.com/GradeK/math/MeasureGeo/MathK_Tall.htm)

**AIR Additional Supports****Background knowledge for students and key academic terms**

At times, students may need access to background knowledge before they can comfortably begin work on a lesson. In this example, the background knowledge for students and key academic terms are included to ensure that students have access to the foundational information required for work in the lesson.

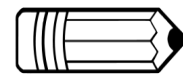
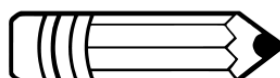
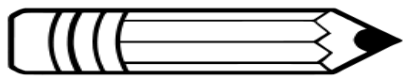
**Concrete and visual models**

For students at the entering, emerging, and transitioning levels of English proficiency, concrete and visual models can make mathematical concepts more apparent and accessible. These models may include manipulatives, illustrations, or other opportunities to have hands-on experiences with the concepts.

Some of the nonessential vocabulary words in this lesson are easily taught. In the application problem, the words *monkey* and *banana* are used, and to best teach these words, the use of photographs, video, or, in the case of the banana, realia, are best. Although the previous lesson used comparatives and superlatives (like tall, taller, tallest), ELLs at the entering, emerging, and transitioning levels may need a review of these terms and how they are related. Review a visual like the one immediately following to reinforce these ideas.

**AIR Routine for Teachers**

- T: Let's take a look at our work from yesterday. Let's look at these drawings of pencils, and use these sentences to compare them. I will read the sentences, and then you will read after me. Ready? This pencil is short.
- S: This pencil is short.
- T: Yes! Next: This pencil is shorter.
- S: This pencil is shorter.
- T: Great! Next: This pencil is shortest.
- S: This pencil is shortest.
- T: Yes! Now let's look at the drawings again: short, shorter, shortest. Which pencil is shortest?



This pencil is short.

This pencil is shorter.

This pencil is shortest.

To add depth and nuance to these terms, comparative phrases, along with visuals, should be introduced. An illustration with key phrases and visual cues (see the example that follows) should be posted in and referred to frequently.

**Structured opportunities to speak with a partner or small group**

Many of the tasks in this unit encourage students to work with a partner. We recommend that ELLs be paired with more proficient English speakers. In addition, there should be some initial training to assist pairs in working together. There are several excellent resources that provide guidance related to academic conversations and activities to help students acquire the skills they need to engage in these conversations. Examples of skills include elaborate and justify, support ideas with examples, build on or

challenge a partner's ideas, paraphrase, and synthesize conversation points. To assist ELLs, we recommend providing students with frames for prompting the skill. For example, for the skill "elaborate and clarify," frames for prompting the skill might be "I am a little confused about the part..."; "Can you tell me more about...?"; "What do you mean by...?"; "Can you give me an example?"; "Could you try to explain it again?" Frames for responding might be "In other words..."; "It's similar to when..."

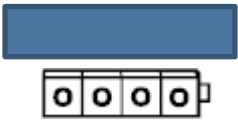

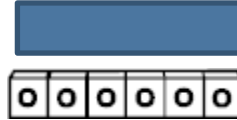
ELLs require multiple opportunities to rehearse their newly learned language skills, and working with partners and small groups of peers is an appropriate way to provide practice opportunities.

Using the phrases that follow, have students compare two objects in the classroom.

### AIR Routine for Teachers

**T:** I am going to compare this marker and this index card. I will use my words from the chart here on the wall. I will hold them side by side. I see that the index card is shorter than the marker. I see that the marker is longer than the index card.

Now you find two objects to compare. Tell your partner what you notice about them. Use the words is shorter than, is longer than, or is the same as to describe what you notice.

 is shorter than	 is longer than	 is the same as; is equal to
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### Background knowledge for students

Not all students may know the word *monkey* and therefore may not know how to draw one. Similarly, not all students may be familiar with the word *banana*. Use photographs (or real bananas, if possible) to ensure that students know the words used in the problem.



monkey



banana

### AIR Routine for Teachers

**T:** This is a monkey. Look at her tail (point to tail). This is a banana. A yellow, delicious banana! Raise your hand if you like bananas.

### Sentence starters

Because communicating in mathematics is essential, ELLs may benefit from the use of sentence starters as ways to express more complex ideas than they may be able to without this scaffold. To scaffold the speaking and writing of ELLs, sentence starters or sentence frames can be very supportive and provide structures for students to use in communicating their thinking.

Use the words from our chart: is longer than or is shorter than. Say, "My monkey's tail is longer than OR is shorter than this banana."

## Common Core Inc. Concept Development

Materials: (S) Popsicle stick and prepared paper bag filled with various items to measure (e.g., pencil, eraser, glue stick, toy car, small block, 12-inch piece of string, marker, child's scissors, crayon, tower of 5 linking cubes) per pair

T: Today you and your partner have a mystery bag! Each of you close your eyes and take something out of the bag. Put the objects on your desk.

T: Here is a popsicle stick. Take one of your objects and compare its length to the popsicle stick. (Select a pair of students to demonstrate. Model and have students repeat correct *longer than* and *shorter than* language if necessary.)  
Student A, what do you notice?

MP.

S: This car is shorter than the popsicle stick.

T: Student B?

S: This pencil is longer than the popsicle stick.

T: Take out another object and compare it to the popsicle stick. Tell your partner what you observe. (Allow time for students to compare the rest of the objects in the bag with the stick.)

T: How could we use the popsicle stick to help us sort these objects?

S: By size! → We could find all of the things that are longer than the length of the stick and the ones that are shorter than the length of the stick.

T: Good idea. Here is a work mat to help you with your sort. (Distribute work mats to students and allow them to begin. During the activity, students may line up objects by size within the sort category. Acknowledge correct examples of this, but do not require it.)

T: What if you put away your popsicle stick and used your toy car instead to help you sort?

S: The sort would come out differently. → This would have to go on the other side!

T: Which objects would you need to move? Let's find out. This time, use your toy car to measure the other things. (Continue the exercise through several iterations, each time sorting with respect to the length of a different object from the bag.)

T: Did anyone notice anything during your sorting?

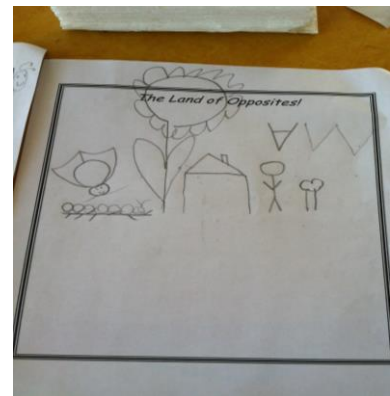
S: It changes every time! → When we used the little eraser to sort, everything else was on the other side. → When we used the string, everything else was on one side. The string was the longest thing.

T: Put your objects back in the bag. Let's use our imaginations to think about length in a



### Notes on Multiple Means for Action and Representation:

Challenge your above grade level students by extending the task for them. Ask them, individually or in teams, to order the objects in their mystery bags from shortest to longest. You can also ask them to be your helpers by finding objects in the classroom that can be added to everyone's mystery bag.



different way in our Problem Set activity.

### Common Core Inc. Problem Set

Students should do their personal best to complete the Problem Set within the allotted 10 minutes (Activity Sheet 1). For some classes, it may be appropriate to modify the assignment by specifying which problems they work on first. Some problems do not specify a method for solving. Students solve these problems using the RDW approach used for Application Problems.

Read the directions carefully to the students. You may wish to use a timer to limit the sketching of each object, leaving a couple of minutes toward the end during which the students may fill in details of their drawing. Circulate during the activity to assess understanding.

**Directions:** Pretend that I am a pirate who has traveled far away from home. I miss my house and family. Will you draw a picture as I describe my home? Listen carefully and draw what you hear.

- Draw a house in the middle of the paper as tall as your finger.
- Now draw my daughter. She is shorter than the house.
- There's a great tree in my yard. My daughter and I love to climb the tree. The tree is taller than my house.
- My daughter planted a beautiful daisy in the yard. Draw a daisy that is shorter than my daughter.
- Draw a branch lying on the ground in front of the house. Make it the same length as the house.
- Draw a caterpillar next to the branch. My parrot loves to eat caterpillars. Of course, the length of the caterpillar would be shorter than the length of the branch.
- My parrot is always hungry and there are plenty of bugs for him to eat at home. Draw a ladybug above the caterpillar. Should the ladybug be shorter or longer than the branch?
- Now draw some more things you think my family would enjoy.
- Show your picture to your partner and talk about the extra things that you drew. Use *longer than* and *shorter than* words when you are describing them.



#### Notes on Multiple Means of Representation:

Modify the directions as necessary depending on the overall ability level of the class. If students seem to tire, curtail the exercise after drawing a few of the objects. If they are very adept at the exercise, give some extra time for the extension activity at the end of the story.

### AIR Additional Supports

#### Background knowledge for teachers

Without sacrificing content, ELLs may at times need instructions and examples provided in language that is more easily accessible and more relevant to their lives.

Because students may have come from communities where actual piracy is ongoing and devastating, the context of being a pirate may not be appropriate for all students. In addition, the pun at the bottom



of the worksheet page, mimicking “pirate” speech (“Home is where the heARRT is”) is not in Standard English and may be interpreted as ridiculing someone’s pronunciation, which is not appropriate for use with ELLs working to learn English.

### **Scaffolded language**

At times, the linguistic complexity of the language impedes student access to the content being taught. To clarify the key concepts and maintain rigor while providing access to the content, it may be necessary to reword some text using present tense, shorter sentences, fewer clauses, and contexts familiar to students.

To provide access to the content of these oral instructions (making a series of comparisons of longer than and shorter than items), using a modified version (see the example that follows) will benefit ELLs at the entering, emerging, and transitioning levels. Note that these instructions will replace the instructions on the original sheet.

### **AIR Routine for Teachers**

(Say to students) I am going on a trip. I will miss my family.

Draw a picture as I tell you about my home so that I can take it with me.

- Draw a house in the middle of the paper. Make it the size of your finger. (Gesture to show the middle of the paper and which finger you want the house to be equal to.)
- Draw my daughter. She is shorter than the house. (Gesture to the word wall card or objects you have been using throughout the lesson while also emphasizing the academic vocabulary is shorter than.)
- Draw a flower that is shorter than my daughter.
- Draw a branch lying on the ground in front of the house. The branch is the same length as my house. (Pair words with gestures and have a visual of a branch.)
- Draw a caterpillar next to the branch. The caterpillar is shorter than the branch. (Pair words with gestures and have a visual of a caterpillar.)
- Draw a ladybug above the caterpillar. (Pair words with gestures and have a visual of a ladybug.) Should the ladybug shorter or longer than the branch? (Note: This will require students to know what a ladybug is, and a photograph might not indicate scale.)
- Now, draw more things you think should be in the picture.
- Show your picture to your partner and tell her or him about the things you drew. Use is longer than and is shorter than to describe your picture to your partner.

## **Common Core Inc. Student Debrief**

The Student Debrief is intended to invite reflection and active processing of the total lesson experience.

Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson. You may choose to use any combination of the questions below to lead the discussion.

- What did you notice when you changed the object you were comparing within our mystery bag activity?

- What did you think about when you were deciding how to draw the ladybug?
- What did you think about when you were deciding how to draw your caterpillar?
- How were the words longer than and shorter than useful when you were telling your partner about your picture?

### AIR Additional Supports

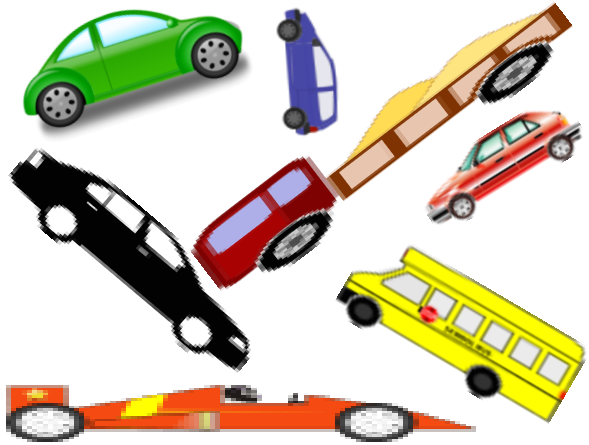
#### Homework scaffolds (Activity Sheets 2 and 3)

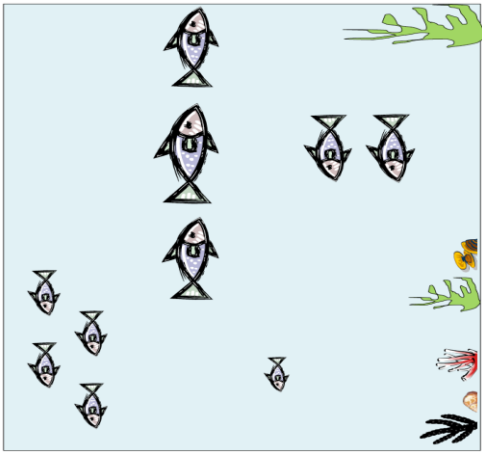
To ensure that ELLs are able to practice their new learning outside school, providing homework scaffolds can be essential. In this example, the homework assignment has been rewritten into more accessible language to ensure that more students have access.

Students may need read-aloud support and modeling to complete some homework assignments, and not all students may have the assigned materials at home (crayons, in this example). Therefore, it may be necessary to provide appropriate materials.

For this assignment, read the homework problem to the students and ensure that they understand what they are supposed to do. If students might not have crayons at home, provide the three crayons (new crayon of any color, blue crayon, and red crayon) in a bag to take home.

### Common Core Inc. Activity Sheets

Activity Sheet 1	Activity Sheet 2
<div data-bbox="203 1144 792 1732" style="border: 2px solid #808080; width: 100%; height: 100%; margin-bottom: 10px;"></div> <p data-bbox="224 1747 748 1787">Home is where the heARRt is, matey.</p>	<p data-bbox="820 1108 1393 1140">Name _____ Date _____</p> <p data-bbox="820 1171 1416 1262">Take out a new crayon. Circle objects with lengths shorter than the crayon blue. Circle objects with lengths longer than the crayon red.</p> <div data-bbox="836 1270 1421 1711" style="text-align: center;">  </div> <p data-bbox="820 1743 1432 1833">On the back of your paper, draw some things shorter than and longer than the crayon. Draw something that is as long as the length of the crayon.</p>

Activity Sheet 3	Activity Sheet 4
<p data-bbox="300 373 357 718"><b>Longer than...</b></p> <hr data-bbox="300 325 743 340"/> <p data-bbox="646 346 703 718"><b>Shorter than...</b></p>	

## Grade 4, Module 5, Lesson 16: Use Visual Models to Add and Subtract Two Fractions With the Same Units

### Overview

The following table outlines the scaffolds that have been added to support ELLs throughout the Common Core Inc. Lesson 16, Use Visual Models to Add and Subtract Two Fractions with the Same Units.

*AIR New Activity* refers to an activity not in the original lesson that AIR has inserted into the original lesson. *AIR Additional Supports* refer to additional supports added to a component already in place in the original lesson. *AIR new activities* and *AIR additional supports* are boxed whereas the text that is in the original lesson is generally not boxed. *AIR Routines for Teachers* are activities that include instructional conversations that take place between teachers and students. In the *AIR Routines for Teachers* the text of the original Common Core Inc. lessons appears in standard black, whereas the AIR additions to the lessons are in green.

Original Component by Common Core, Inc.	AIR Additional Supports	AIR New Activities
<i>None included</i>		Cueing Key academic vocabulary
Count by Equivalent Fractions	Concrete and visual models Teacher modeling and explanation Structured opportunities to speak with a partner or small group	
Compare Fractions	Key academic vocabulary Recording and processing key ideas	
Application Problem	Sentence starters Concrete and visual models	
Concept Development	Structured opportunities to speak with a partner or small group Background knowledge for teachers Background knowledge for students Key academic vocabulary	
Student Debrief	Structured opportunities to speak with a partner or smaller group	
Exit Ticket	Key academic vocabulary Cueing	
Homework	Homework scaffolds	

## AIR Lesson Introduction

### AIR New Activities

#### Background knowledge for teachers

Background knowledge for teachers is provided as a way to help teachers become more familiar with the educational contexts their students may have experienced before beginning in U.S. schools. This is important to help teachers tailor instruction and assessments to ensure that all students are appropriately challenged and supported.

School systems outside the United States may not emphasize operations with fractions as we do in the United States, and instruction may not involve fraction computation until secondary school. Students unfamiliar with fractions should have opportunities to make connections between the area model, the set model, and the distance model for fractional parts.

#### Cueing

The lesson opens with cueing to provide an anticipatory overview for students (in the form of an objective and agenda in student-friendly language) and to provide an initial introduction to the key vocabulary of the lesson. This provides an advance schema for the students and allows them to begin to anticipate how the new information will connect to previous learning. The purpose of this cueing is to establish what the lesson will be about and to help students know where to focus their attention throughout the lesson. Further, this lesson begins with clear introductions to key academic vocabulary so that students will be able to more quickly access the content of the lesson.

Because this section is not included in the original lesson, subsequent times allotted for activities have been modified.

#### Key academic vocabulary

Introducing new vocabulary should be explicit and should take into account words that are homophones (like *sum* and *some*), words that have multiple functions (like the word *number* which can be used as a noun, a verb, and adjective), and words that are often difficult for students to accurately hear or pronounce (like *eighths*). Students should be provided with structures to record their new vocabulary, such as graphic organizers, a specific note-taking format, or a student-created illustrated dictionary.

ELLs should be given opportunities to review vocabulary to which they have been exposed but may not have committed to memory. For example, the modified version of this lesson opens with a brief review of the word *compare* from the previous day's lesson. If time permits, students could review vocabulary with partners using flash cards or a folded graphic organizer they have created. They could review vocabulary by lesson or in other ways, such as by grammatical form (nouns, verbs, phrases). The cards or folded graphic organizer could have the vocabulary word and perhaps an illustration on the front and the back could contain a definition, a first-language translation, an exemplary sentence, and questions that would engage the students in discussion about the words.

ELLs should have structured opportunities to use this key academic vocabulary—both new terms and previously learned terms—across all four modalities (speaking, reading, writing, and listening) each day.

### AIR Routine for Teachers

Introduce objectives, student outcomes, and key vocabulary for the lesson. Display the standard associated with this lesson. Write on board and read aloud:

T: I will use pictures and manipulatives to show how to add and subtract fractions with the same units.

T: In our last lesson, you compared fractions to see what was similar and what was different. In

this lesson, you will combine or add fractions together (model bringing hands together), and you will also separate or subtract fractions [model bringing hands apart]. [Have students follow along with their hands: “When we combine or add, we put them together [bring hands together]; when we separate or subtract, we move them apart [bring hands apart]. What are other ways we might talk about adding or subtracting? What other words might we use? (Allow students to turn-and-talk and then share responses.)

T: Today, we will be using some of the same units we have used before, like halves, thirds, and fourths. What are some other units we might talk about today? (Elicit other fractional parts from students.)

Introduce the day’s agenda by posting it and reviewing it orally while indicating each activity for the day’s lesson.

## Common Core Inc. Count by Equivalent Fractions

This activity builds fluency with equivalent fractions. The progression builds in complexity. Work the students up to the highest level of complexity in which they can confidently participate.

### AIR Additional Supports

#### Concrete and visual models

For students at the entering, emerging, and transitioning levels of English proficiency, concrete and visual models can make mathematical concepts more apparent and accessible. These models may include manipulatives, illustrations, or other opportunities to have hands-on experiences with the concepts. Providing concrete and visual models is important because these approaches illustrate key content in ways that are meaningful and clear for students.

#### Teacher modeling and explanation

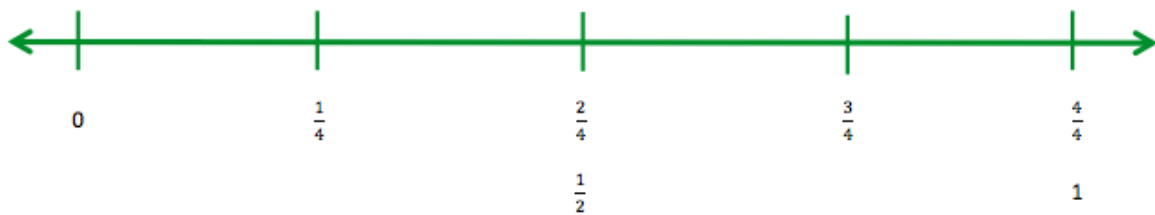
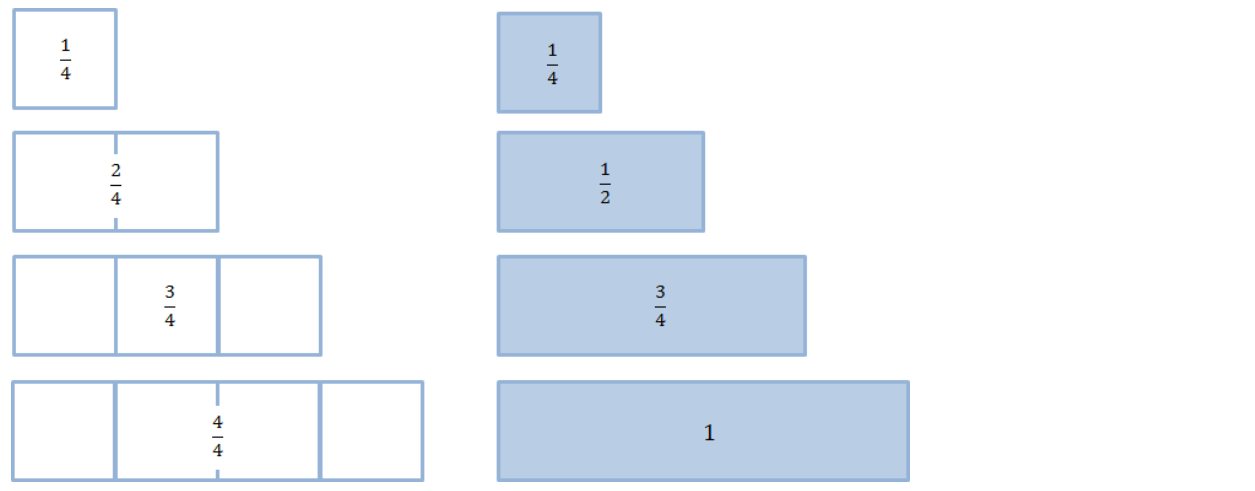
Teacher explanation and modeling of thought processes, of the manner in which lesson activities should be carried out, and of high-quality responses will be particularly beneficial for ELLs because explanation and examples enhance comprehension. We suggest explanation and modeling be used to support students before they are struggling, with teachers clearly explaining each task and modeling an expected student response. For example, when counting around the room by fourths, the teacher should carefully pronounce *fourths* to provide an accurate auditory imprint of the sounds in the word for students. In the student debrief at the end, the teacher should refer to the sample sentence starters and modeled using them to guide the conversation. Teachers should model all mathematical discourse that students will be using in their own collaborations with peers and in writing.

#### Structured opportunities to speak with a partner or small group

Many of the tasks in this unit encourage students to work with a partner. We recommend that ELLs be paired with more proficient English speakers. In addition, there should be some initial training to assist pairs in working together. There are several excellent resources that provide guidance related to academic conversations and activities to help students acquire the skills they need to engage in these conversations. Examples of skills include elaborate and justify, support ideas with examples, build on or challenge a partner’s ideas, paraphrase, and synthesize conversation points. To assist ELLs, we recommend providing students with frames for prompting the skill. For example, for the skill “elaborate and clarify,” frames for prompting the skill might be “I am a little confused about the part...”; “Can you tell me more about...?”; “What do you mean by...?”; “Can you give me an example?”; “Could you try to explain it again?” Frames for responding might be “In other words...”; “It’s similar to when...”

### AIR Routine for Teachers

ELLs, particularly those at the entering and emerging levels, may still be developing language to describe the concept of equivalent fractions. To support students' understanding and language development, pair the numerical progression in this portion of the lesson with visual models like the number line, tape diagrams, and the or area model.



T: Starting at zero, count by ones to 8.

S: 0,

T: 8

0	1	2	3	4	5	6	7	8
$\frac{0}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$
0	1	2	3	4	5	6	7	1 whole
$\frac{0}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1 whole
0	1	2	3	4	5	6	7	1 whole
$\frac{0}{8}$	$\frac{1}{4}$	$\frac{2}{8}$	$\frac{3}{4}$	$\frac{4}{8}$	$\frac{5}{4}$	$\frac{6}{8}$	$\frac{7}{4}$	1 whole

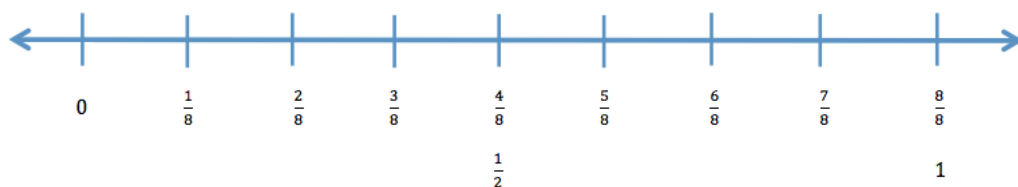
1, 2, 3, 4, 5, 6, 7, 8.

Starting at 0 eighths, count by 1 eighths to 8 eighths.

(Write as students count.)

$\frac{0}{8}$ ,  $\frac{1}{8}$ ,  $\frac{2}{8}$ ,  $\frac{3}{8}$ ,  $\frac{4}{8}$ ,  $\frac{5}{8}$ ,  $\frac{6}{8}$ ,  $\frac{7}{8}$ ,  $\frac{8}{8}$

Repeat, and show on number line (divided into eighths).



T: (Point to  $\frac{8}{8}$ .) 8 eighths is the same as 1 of what unit?

S: 1 whole.

T: (Beneath  $\frac{8}{8}$ , write *1 whole*.) Count by 1 eighths from zero to 1. This time, when you come to 1

whole, say “1 whole.” Try not to look at the board.

S:  $0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1$  whole.

T: (Point to  $\frac{4}{8}$ .) 4 eighths is the same as 1 of what unit?

S:  $\frac{1}{2}$ .

T: (Beneath  $\frac{4}{8}$ , write  $\frac{1}{2}$ .) Count by 1 eighths again. This time, convert to  $\frac{1}{2}$  and 1 whole. Try not to look at the board.

For ELLs, use the term *simplify* instead of the word *convert* (which is related to units of measure and is taught elsewhere in the curriculum).

S:  $\frac{0}{8}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1$  whole.

T: What other fractions can we simplify?

S:  $\frac{2}{8}$  and  $\frac{6}{8}$ .

T: (Point to  $\frac{2}{8}$ .) What’s 2 eighths simplified?

S:  $\frac{1}{4}$ .

T: (Beneath  $\frac{2}{8}$ , write  $\frac{1}{4}$ . Point to  $\frac{6}{8}$ .) What’s  $\frac{6}{8}$  simplified?

S:  $\frac{3}{4}$ .

T: (Beneath  $\frac{6}{8}$ , write  $\frac{3}{4}$ .) Count by 1 eighths again. This time, convert to  $\frac{1}{4}$  and  $\frac{3}{4}$ . Try not to look at the board.

S:  $\frac{0}{8}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1$  whole.

Direct students to count back and forth from 0 to 1 whole, occasionally changing directions.



**AIR Additional Supports****Teacher modeling and explanation**

ELLs may not initially understand instructions given by the teacher and may benefit from modeling to help reinforce the directions.

To further reinforce these ideas for ELLs, after practicing choral counting, engage the students in a count around the room. Counting with the students and beginning with fourths, the first student (with the teacher) says “one fourth.” The next student says “two fourths.” The third student would say “three fourths,” and the fourth student would say “four fourths.” Pause before student 5 and ask the class what the next student would say, and then have the student respond, “one and one fourth”. Continue around the entire class until every student has a chance to count-on. Allow the students to naturally include equivalent fractions if they have the knowledge that “two fourths” is the same as “one half.” Then have students repeat this counting, but begin with a different student. In this activity, multiple students are responsible for participating. Students may want to use their personal whiteboards to write themselves notes, such as  $\frac{4}{4}=1$  whole.

**Structured opportunities to speak with a partner or small group**

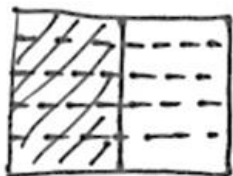
Repeated opportunities to practice communicating mathematical thinking are essential, and to reduce the stress of speaking in front of the whole class, speaking with a partner or small group can be a more appropriate venue for ELLs.

After counting around the room, to reinforce the concept of the unit and the whole, ask the class, “*How many students did we need to count to one whole when counting in fourths?*” Have students turn to their assigned partner to discuss for 15 seconds. Have students write their responses on their whiteboards and hold them up.

It would be beneficial to repeat this activity daily with different units to connect the idea of supporting students’ fluency and understanding of equivalent fractions while building their oral skills in pronouncing fractions.

**Common Core Inc. Compare Fractions**

Materials: (S) Personal white boards



$$\frac{1}{2}$$



$$\frac{2}{5}$$

**AIR Additional Supports****Key academic vocabulary and recording and processing key ideas**

Although fluent speakers of English may already be familiar with key academic vocabulary, ELLs will benefit from instruction to explain and add depth to meanings. Further, ELLs may need clear parameters for recording and processing their ideas, which may not be familiar practices. Recording main concepts and ideas can deepen understanding and increase retention, positioning ELLs to access the complex ideas and language used in this lesson.

**AIR Routine for Teachers**

Introduce the term *partition*.

T: Here is a new word we will be using. It is the verb partition. Look at the beginning of this word. We can see the word part at the beginning. The verb partition means to divide into parts. Again, the verb partition means to divide into parts. Watch as I draw this rectangle and then partition it into two halves. Tell your neighbor what the verb partition means. Now write partition in your personal glossary. Write “verb: to divide into parts.” I will add the word partition to our word wall.

Note: This fluency activity reviews G4–M5–Lesson 15.

T: On your boards, draw two area models. (Allow students time to draw.)

T: (Write  $\frac{1}{2}$ .) Partition (which is the same as divide) your first diagram into an area model that shows  $\frac{1}{2}$ . Then, write  $\frac{1}{2}$  beneath it.

S: (Partition first area model into 2 equal units. Shade one unit. Write  $\frac{1}{2}$  beneath it.)

T: (Write  $\frac{1}{2}$  —  $\frac{2}{5}$ .) Partition your second area model to show  $\frac{2}{5}$ . Then, write  $\frac{2}{5}$  beneath it.

S: (Partition second area model into 5 equal units. Shade 2 units. Write  $\frac{2}{5}$  beneath the shaded area.)

T: Partition the area models so that both fractions have common denominators.

S: (Draw dotted lines through the area models.)

T: Write a greater than, less than, or equal sign to compare the fractions.

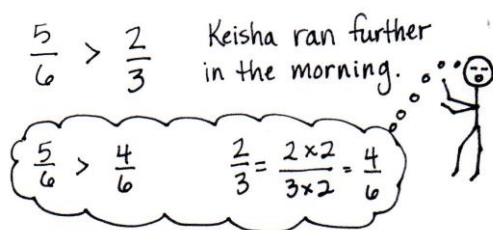
S: (Write  $\frac{1}{2} > \frac{2}{5}$ .)

Continue the process, comparing  $\frac{1}{5}$  and  $\frac{3}{10}$ ,  $\frac{1}{4}$  and  $\frac{5}{8}$ , and  $\frac{1}{3}$  and  $\frac{3}{4}$ .

### Structured opportunities to speak with a partner or small group

Focus a discussion on the two students' strategies to compare  $\frac{1}{2}$  to  $\frac{2}{5}$  using a visual model of the students' choice. Ask, "How did \_\_\_'s strategy help you compare  $\frac{1}{2}$  to  $\frac{2}{5}$ ?" "Explain to your neighbor which strategy was easier for you. Explain why it was easier for you." Then students apply their new understandings to compare  $\frac{1}{4}$  to  $\frac{5}{8}$ . Ask, "Which problem was easier to compare visually, with pictures?" Focus responses on benchmark fractions (fourths and eighths).

### Common Core Inc. Application Problem



### AIR Additional Supports

#### Clarifying language, sentence starters, and visuals

ELLs benefit from language that has been clarified, sentence starters and visuals

**AIR Routine for Teachers****Clarifying language**

Keisha ran  $\frac{5}{6}$  mile in the morning and  $\frac{2}{3}$  mile in the afternoon. **Which distance was longer?** Did Keisha run farther in the morning or in the afternoon? Solve independently. Share your solution with your partner. Did your partner solve the problem in the same way or a different way? Explain.

**Sentence starters**

To scaffold the speaking and writing of ELLs, sentence starters or sentence frames can be very supportive and provide structures for students to use in communicating their thinking. Using sentence starters is a simple way to help ELLs structure their thinking and create meaningful sentences with increasing sophistication.

First, I \_\_\_\_\_. Then, I \_\_\_\_\_. Next, I \_\_\_\_\_. Finally, I \_\_\_\_\_. Model it with another example so that students have an example of how to use it.

**Visual support**

Students may need a visual to support their understanding of  $\frac{5}{6}$  to  $\frac{2}{3}$ . It would be beneficial to show a variety of solutions. Use a number line or tape diagram to show distance.

*Note:* This application problem builds on the concept development of G4–M5, Lessons 14 and 15, where students learned to compare fractions with unrelated denominators by finding common units.

**Common Core Inc. Concept Development**

Materials: (S) Personal white board, Practice Sheet

**Problem 1: Solve for the difference using unit language and a number line**

T: (Project 5 – 4.) Solve. Say the number sentence using units of ones.

S: 5 ones – 4 ones = 1 one.

T: Say the number sentence if the unit is dogs. (For ELLs, use something concrete in the classroom instead of dogs, like paper clips or unit cubes.)

S: 5 dogs – 4 dogs = 1 dog.

T: Say the number sentence if the unit is meters.

S: 5 meters – 4 meters = 1 meter.

T: Say the number sentence if the unit is sixths.

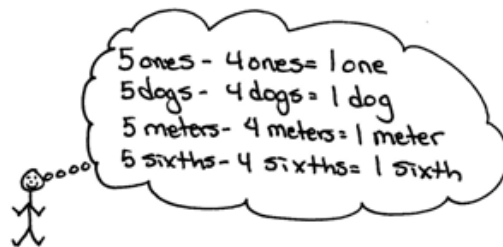
S: 5 sixths – 4 sixths = 1 sixth.

T: Let's show that 5 sixths – 4 sixths = 1 sixth.

T: (Project number line with endpoints 0 and 1, partitioned into sixths.) Make tick marks on the first number line on your Practice Sheet to make a number line with endpoints 0 and 1 above the number line. Partition (*divide*) the number line into sixths. (See illustration below.)

T: Draw a point at 5 sixths. Put the tip of your pencil on the point. Count backwards to subtract 4 sixths.

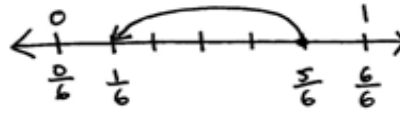
T: Move your pencil and count back with me as we subtract. 1 sixth, 2 sixths, 3 sixths, 4



sixths.

S: 1 sixth!

T: Draw one arrow above the number line to model  $\frac{5}{6} - \frac{4}{6}$ . (Demonstrate.) Tell me the sentence.



$$\frac{5}{6} - \frac{4}{6} = \frac{1}{6}$$

subtraction

S:  $\frac{5}{6} - \frac{4}{6} = \frac{1}{6}$ .

Repeat with  $\frac{7}{8} - \frac{3}{8}$ .

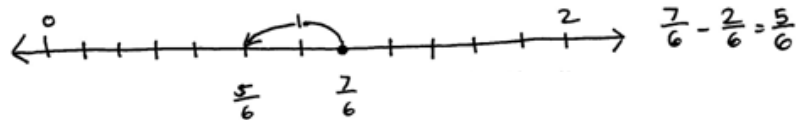
T: Solve for 7 sixths – 2 sixths. Work with a partner. Use the language of units and subtraction.

S: 7 sixths – 2 sixths = 5 sixths. → I know 7 ones minus 2 ones is 5 ones. I can subtract sixths like I subtract ones.  $\frac{7}{6} - \frac{2}{6} = \frac{5}{6}$ .

T: Discuss with your partner how to draw a number line to represent this problem.

S: We partition it *or divide it* like the first problem and draw the arrow to subtract. → But,  $\frac{7}{6}$  is more than 1 whole. 6 sixths is equal to 1. We have 7 sixths. → Let's make the number line with endpoints 0 and 2.

T: Label the endpoints 0 and 2. Partition the number line into sixths. Subtract.



S: On the number line, we started at 7 sixths and then went back 2 sixths. The answer is 5 sixths. →  $\frac{7}{6} - \frac{2}{6} = \frac{5}{6}$ .

Repeat with  $\frac{7}{4} - \frac{5}{4}$ .



### Notes On Multiple Means Of Representation:

Be sure to articulate the ending digraph /th/ to distinguish *six* from *sixth* for English language learners. Coupling spoken expressions with words or models may also improve student comprehension. For example, write out  $5 \text{ sixths} - 4 \text{ sixths} = 1 \text{ sixth}$ .



### Notes on Multiple Means of Engagement:

Students working above grade level and others may present alternative subtraction strategies, such as counting up rather than counting down to solve  $\frac{7}{6} - \frac{5}{6}$ . Though not introduced in this lesson, the appropriate use of these strategies is desirable and will be introduced later in the module.

## Air Additional Supports

Structured opportunities to speak with a partner or small group

During these tasks, ELLs should be paired with English-proficient peers to facilitate engagement in academic conversations in English. It also would be beneficial to reinforce the concept of equivalency by always showing multiple representations of the problems.

This lesson assumes that students know that the right side of the number line represents larger numbers than those on the left. This might not be known or clear to all students, particularly those who may have literacy in a language that reads right to left.

### Background knowledge for students

Because ELLs may have attended schools outside the United States, or may have not fully learned the content from previous grades, building background knowledge for students can be an essential part of scaffolding. In the previous lesson, students were asked to compare two fractions, noting similarities and differences. Because the word *difference* was recently used with a meaning unlike that intended here, briefly review that “finding the difference” or “solving for the difference” means to subtract in this context.

## Problem 2: Decompose to record a difference greater than 1 as a mixed number

### AIR Additional Supports

#### Key academic vocabulary

Although students may have heard the verb *decompose* in a previous lesson, review the definition for clarity.

### AIR Routine for Teachers

**T:** When we use the verb decompose in mathematics class, it means to break something, like a number into smaller pieces. For example, we can decompose the number four by saying that it is two plus two.

**T:** (Display  $10 \text{ sixths} - 2 \text{ sixths}$ .) Solve in unit form and write a number sentence using fractions.

**S:**  $10 \text{ sixths} - 2 \text{ sixths}$  is  $8 \text{ sixths}$ .  $\frac{10}{6} - \frac{2}{6} = \frac{8}{6}$ .

**T:** Use a number bond to decompose  $\frac{8}{6}$  into the whole and fractional parts.

Students draw number bond as pictured to the right.

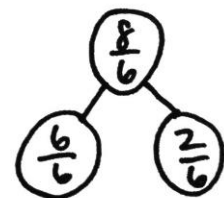
**T:**  $\frac{6}{6}$  is the same as...?

**S:** 1 whole.

**T:** We can rename  $\frac{8}{6}$  as a **mixed number**,  $1\frac{2}{6}$ , using a whole number and fractional parts. Repeat with  $9 \text{ fifths} - 3 \text{ fifths}$ .

$$\frac{9}{5} - \frac{3}{5} = \frac{6}{5} = 1\frac{1}{5}$$

$\frac{5}{5}$     $\frac{1}{5}$



**Problem 3: Solve for the sum using unit language and a number line.**

T: Look back at the first example. (Point to the number line representing 5 sixths – 4 sixths.) Put your finger on 1 sixth. To 1 sixth, let's add the 4 sixths that we took away.

T: Count as we add. 1 sixth, 2 sixths, 3 sixths, 4 sixths. Where are we now?

S: 5 sixths.

T: What is 1 sixth plus 4 sixths?

S: 5 sixths.

T: Let's show that on the number line.

Model with students as shown to the right.

T: 1 one plus 4 ones is...?

S: 5 ones.

T: 1 apple plus 4 apples is...?

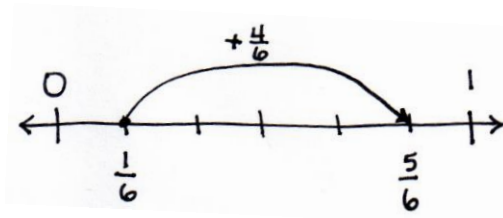
S: 5 apples.

T: 1 sixth plus 4 sixths equals?

S: 5 sixths.

Repeat with  $\frac{2}{8} + \frac{3}{8} = \frac{5}{8}$ .

$$\frac{1}{6} + \frac{4}{6} = \frac{5}{6}$$

**Problem 4: Decompose to record (or write down) a sum greater than 1 as a mixed number****AIR Additional Supports****Key academic vocabulary**

Remember that the noun *sum* sounds just like the noun *some*. Review the definition for clarity.

**AIR Routine for Teachers**

**T:** In our mathematics class, the word sum is the result of addition. It is the answer after we have added.

T: (Display 5 fourths + 2 fourths.) Solve in unit form, and write a number sentence using fractions.

S: 7 fourths.  $\frac{5}{4} + \frac{2}{4} = \frac{7}{4}$ .

T: Use a number bond to decompose  $\frac{7}{4}$  into the whole and some parts.

Students draw number bond as pictured to the right.

T:  $\frac{4}{4}$  is the same as...?

S: 1 whole.

T: We can rename  $\frac{7}{4}$  as a mixed number,  $1\frac{3}{4}$ .

Repeat with 6 sixths + 4 sixths.

$$5 \text{ fourths} + 2 \text{ fourths} = 7 \text{ fourths}$$

$$\frac{5}{4} + \frac{2}{4} = \frac{7}{4} = 1\frac{3}{4}$$

$\wedge$   
 $\frac{4}{4} \quad \frac{3}{4}$

## Common Core Inc. Problem Set

Students should do their personal best to complete the Problem Set within the allotted 10 minutes. For some classes, it may be appropriate to modify the assignment by specifying which problems they work on first. Some problems do not specify a method for solving. Students solve these problems using the RDW approach used for Application Problems.

Lesson 16 Problem Set 4•5

Name Jack Date \_\_\_\_\_

1. Solve.

a. 3 fifths - 1 fifth = 2 fifths      b. 5 fifths - 3 fifths = 2 fifths

c. 3 halves - 2 halves = 1 half      d. 6 fourths - 3 fourths = 3 fourths

2. Solve.

a.  $\frac{5}{6} - \frac{2}{6} = \frac{3}{6}$       b.  $\frac{6}{8} - \frac{4}{8} = \frac{2}{8}$

c.  $\frac{3}{10} - \frac{3}{10} = \frac{0}{10}$       d.  $\frac{5}{5} - \frac{4}{5} = \frac{1}{5}$

e.  $\frac{5}{4} - \frac{4}{4} = \frac{1}{4}$       f.  $\frac{5}{4} - \frac{3}{4} = \frac{2}{4}$

3. Solve. Use a number bond to show how to convert the difference to a mixed number. Problem 3a has been completed for you.

a.  $\frac{12}{8} - \frac{3}{8} = \frac{9}{8} = 1\frac{1}{8}$       b.  $\frac{12}{6} - \frac{5}{6} = \frac{7}{6} = 1\frac{1}{6}$

c.  $\frac{9}{5} - \frac{3}{5} = \frac{6}{5} = 1\frac{1}{5}$       d.  $\frac{14}{8} - \frac{3}{8} = \frac{11}{8} = 1\frac{3}{8}$

e.  $\frac{8}{4} - \frac{2}{4} = \frac{6}{4} = 1\frac{2}{4}$       f.  $\frac{15}{10} - \frac{3}{10} = \frac{12}{10} = 1\frac{2}{10}$

COMMON CORE Lesson 16: Use visual models to add and subtract two fractions with same units. 5/3/13  
engage<sup>ny</sup> 5.D.8

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Name \_\_\_\_\_ Date \_\_\_\_\_

1. Solve.

a. 3 fifths  $-$  1 fifth = \_\_\_\_\_

b. 5 fifths  $-$  3 fifths = \_\_\_\_\_

c. 3 halves  $-$  2 halves = \_\_\_\_\_

d. 6 fourths  $-$  3 fourths = \_\_\_\_\_

2. Solve.

a.  $\frac{5}{6} - \frac{3}{6}$

b.  $\frac{5}{6} - \frac{3}{6}$

c.  $\frac{3}{10} - \frac{3}{10}$

d.  $\frac{5}{5} - \frac{4}{5}$

e.  $\frac{5}{4} - \frac{4}{4}$

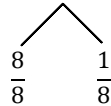
f.  $\frac{5}{4} - \frac{3}{4}$

3. Solve. Use a number bond to show how to convert the difference to a mixed number.

Problem (a) has been completed for you.

a.  $\frac{12}{8} - \frac{3}{8} = \frac{9}{8} = 1 \frac{1}{8}$

b.  $\frac{12}{8} - \frac{3}{8} =$



c.  $\frac{9}{5} - \frac{3}{5}$

d.  $\frac{14}{8} - \frac{3}{8}$

e.  $\frac{8}{4} - \frac{2}{4}$

f.  $\frac{15}{10} - \frac{3}{10}$

4. Solve. Write the sum in unit form.

a. 2 fourths + 1 fourth = \_\_\_\_\_

b. 4 fifths + 3 fifths = \_\_\_\_\_

5. Solve.

a.  $\frac{2}{8} + \frac{5}{8}$

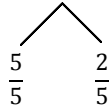
b.  $\frac{4}{12} + \frac{5}{12}$

6. Solve. Use a number bond to decompose the sum. Record your final answer as a mixed number.

Problem (a) has been completed for you.

a.  $\frac{3}{5} + \frac{4}{5} = \frac{7}{5} = 1 \frac{2}{5}$

b.  $\frac{4}{4} + \frac{3}{4}$



c.  $\frac{6}{9} + \frac{6}{9}$

d.  $\frac{7}{10} + \frac{6}{10}$

e.  $\frac{5}{6} + \frac{7}{6}$

f.  $\frac{9}{8} + \frac{5}{8}$

7. Solve. Then use a number line to model your answer.

a.  $\frac{7}{4} - \frac{5}{4}$

b.  $\frac{5}{4} + \frac{2}{4}$

## AIR Additional Supports

### Key academic vocabulary

For ELLs, the term *number bond* may not be familiar from previous lessons, and students may need instruction or a reminder of what it means. Focus student attention on the verb *convert*, which in this case, means to rewrite a fraction greater than 1 as a mixed number.

Consider identifying one or two key problems from each section of the worksheet for ELLs and allowing them to show their thinking in different ways (with number lines, pattern blocks, etc.) Students would benefit from working with a partner to compare strategies.

## Common Core Inc. Student Debrief

The Student Debrief is intended to invite reflection and active processing of the total lesson experience. Invite students to review their solutions for the Problem Set. They should check work by comparing answers with a partner before going over answers as a class. Look for misconceptions or misunderstandings that can be addressed in the Debrief. Guide students in a conversation to debrief the Problem Set and process the lesson.

You may choose to use any combination of the questions below to lead the discussion.

- How do Problems 1(a) through (d) and 4(a) through (b) help you to understand how to subtract or add fractions?
- In Problem 3 and Problem 6 of the Problem Set, how do the number bonds help to decompose the fraction into a **mixed number**?
- Why would we want to name a fraction greater than 1 using a mixed number?
- How is the number line helpful in showing how we can subtract and add fractions with like units?
- How were number bonds helpful in showing how we can rename fractions greater than 1 as 1 whole and a fraction?
- How would you describe to a friend how to subtract and add fractions with like units?

COMMON CORE MATHEMATICS CURRICULUM Lesson 16 Problem Set 4•5

4. Solve. Write the sum in unit form.

a. 2 fourths + 1 fourth = 3 fourths      b. 4 fifths + 3 fifths = 7 fifths

5. Solve.

a.  $\frac{2}{8} + \frac{5}{8} = \frac{7}{8}$       b.  $\frac{4}{12} + \frac{5}{12} = \frac{9}{12}$

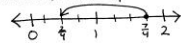
6. Solve. Use a number bond to decompose the sum. Record your final answer as a mixed number. Problem (a) has been completed for you.

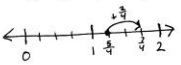
a.  $\frac{2}{5} + \frac{4}{5} = \frac{6}{5} = 1\frac{1}{5}$       b.  $\frac{4}{4} + \frac{3}{4} = \frac{7}{4} = 1\frac{3}{4}$

c.  $\frac{6}{9} + \frac{6}{9} = \frac{12}{9} = 1\frac{2}{3}$       d.  $\frac{7}{10} + \frac{6}{10} = \frac{13}{10} = 1\frac{3}{10}$

e.  $\frac{5}{6} + \frac{7}{6} = \frac{12}{6} = 2$       f.  $\frac{3}{8} + \frac{5}{8} = \frac{8}{8} = 1$

7. Solve. Then use a number line to model your answer.

a.  $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$       

b.  $\frac{3}{4} + \frac{1}{4} = 1$       

COMMON CORE Lesson 16: Use visual models to add and subtract two fractions with same units. 11/14/13 engage<sup>ny</sup> 5.D.9

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**AIR Additional Supports****Structured opportunities to speak with a partner or small group**

ELLs at the entering, emerging, or transitioning levels may need additional language support to fully participate in the class discussion.

**Sentence frames**

Post the following sentence frames and rehearse their use by reading through them as a class and practicing different terms that could fill the blanks.

- The \_\_\_\_\_ model helps me add fractions because \_\_\_\_\_.
- Number bonds help to decompose fractions into mixed numbers because \_\_\_\_\_.
- When adding fractions, it is important to remember to \_\_\_\_\_.

**Common Core Inc. Exit Ticket**

After the Student Debrief, instruct students to complete the Exit Ticket. A review of their work will help you assess the students' understanding of the concepts that were presented in the lesson today and plan more effectively for future lessons. You may read the questions aloud to the students.

**AIR Additional Supports****Key academic vocabulary and cueing**

Clarifying key academic vocabulary and cueing ELLs so they are familiar with lessons objectives will support ELLs.

**AIR Routine for Teachers**

**T:** In today's lesson, you combined or added fractions together, and you also separated or subtracted fractions. You used pictures and manipulatives to show this. Here are the activities we did today: [review the agenda posted at the beginning of class].

To reinforce the objective of the lesson (using visual models), require students to include a visual representation along with their number bonds.

1. Solve. Use a number bond to decompose the difference. Record your final answer as a mixed number. Use a visual representation (like a number line) to show your thinking.

$$\frac{16}{9} - \frac{5}{9}$$

2. Solve. Use a number bond to decompose the sum. Record your final answer as a mixed number. Use a visual representation (like a number line) to show your thinking.

$$\frac{5}{12} + \frac{10}{12}$$

## Homework

### **AIR Additional Supports**

#### **Homework scaffold**

Providing scaffolded homework assignments can provide the teacher with information about the level of depth to which students understand content. Further, it can be useful to echo the format of the lesson in the homework, as is included in the addition that follows, instructing students to use a number line to illustrate their thinking.

You may scaffold the assignment to allow students to focus on a smaller number of problems in greater depth and provide the teacher with formative data on what students understand. Teachers may choose to assign one or two of each type of problem.

## Grade 8, Module 3, Lesson 6: Proofs of Laws of Exponents

### Overview

The following table outlines the scaffolds that have been added to support ELLs throughout the Common Core Inc. Lesson 6, Proofs of Laws of Exponents.

*AIR New Activity* refers to an activity not in the original lesson that AIR has inserted into the original lesson. *AIR Additional Supports* refer to additional supports added to a component already in place in the original lesson. *AIR new activities* and *AIR additional supports* are boxed whereas the text that is in the original lesson is generally not boxed. *AIR Routines for Teachers* are activities that include instructional conversations that take place between teachers and students. In the *AIR Routines for Teachers* the text of the original Common Core Inc. lessons appears in standard black, whereas the AIR additions to the lessons are in green.

Original Component by Common Core, Inc.	AIR Additional Supports	New Activities
<i>None included</i>	Cueing	Cueing
None included		Key academic vocabulary Graphic organizers or foldables
Socratic discussion (first)	Clarification of key concepts Structured opportunities for students to speak with a partner or small group	Homework assignment for previous day
Exercises 1–3	Clarification of key concepts	
Socratic discussion (second)	Teacher background knowledge Key academic vocabulary Clarification of key concepts	
Socratic discussion (third)	Clarification of key concepts	
Exercise 4	<i>No changes suggested</i>	
Socratic discussion (fourth)	Clarification of key concepts Structured opportunities for students to speak with a partner or in small groups Teacher background knowledge	
Closing	Clarification of key concepts Graphic organizer or foldables	
Exit ticket	Clarification of key concepts Graphic organizer or foldables	
Problem set	Clarification of key concepts	

## Common Core, Inc. Lesson Introduction

### Student Outcomes

- Students extend the previous laws of exponents to include all integer exponents.
- Students base symbolic proofs on concrete examples to show that  $(x^b)^a = x^{ab}$  is valid for all integer exponents.

### Lesson Notes

This lesson is not designed for all students, but for those who would benefit from a lesson that enriches their existing understanding of the Laws of Exponents. For that reason, this is an optional lesson that can be used with students who have demonstrated mastery over concepts in Topic A.

#### AIR Additional Supports

##### Background knowledge for teachers

It can be useful for teachers to consider certain aspects of the content as related to previous experiences ELLs may have had. This background information for teachers can help teachers consider which scaffolds for instruction and assessment may be most appropriate for each student.

To successfully participate in the Socratic discussion described in this lesson, students at the entering, emerging, transitioning, and expanding levels of English proficiency must have access to a scaffolded version the text (noted in what follows) and information at an appropriate reading level in advance to prepare for the discussion. Therefore, the lesson described here will require a homework assignment in the previous class meeting to allow students time to prepare for the Socratic discussion.

#### AIR New Activity

##### Cueing, introduction of objectives, student outcomes, and key vocabulary for lesson

Cueing is a way for teachers to orient students to the content and key terms to be used in the upcoming lesson. It can help students know which facets of the lesson are most crucial and which parts of classroom discourse or interaction may be less relevant to the objective for the lesson. At the beginning of the lesson, we include a clear focus on stating the objectives for the day, communicated in student-friendly language. This provides an advance schema for the students and allows them to begin to anticipate how the new information will connect to previous learning.

#### AIR Routine for Teachers

Display the standard associated with this lesson: 8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

Write on board and read aloud:

*I will learn about laws of exponents when the exponents are integers (all positive and negative counting numbers). I will write a proof to show that  $(x^b)^a = x^{ab}$  works for all integers.*

**T:** In this lesson you will learn about more laws of exponents related to integer exponents. You also will write a proof to show that  $(x^b)^a = x^{ab}$  is valid for all integer examples (and not just whole numbers).

**T:** Remember the term exponent that we have been using in this unit. Remember that the exponent of a number tells us how many times to use that number in multiplication. So if I write  $4^5$ , this means I multiply 4 times itself five times. Tell your partner an example of an

exponent and what it means.

- T:** Remember the term integer that we have been using in this unit. Remember that integers are whole numbers that are positive, negative, or zero. Fractions and decimals are not integers. Some examples of integers are four, negative four, and zero. Tell your partner three more integers.

Introduce the day's agenda by posting it and reviewing it orally while indicating each activity for the day's lesson.

## AIR New Activities

### Key academic vocabulary

Specifically teaching key vocabulary is essential for ELLs because a working knowledge of these terms is necessary for comprehending the lesson. Graphic organizers or foldables are a useful way to help students organize information for rehearsal, review, and recall.

In addition, ELLs should be given opportunities to review vocabulary they have already learned. For example, students could review vocabulary with partners using flash cards or the folded graphic organizer they have created. They could review vocabulary by lesson or in other ways, such as by grammatical form (nouns, verbs, phrases). The cards could have the vocabulary word and perhaps an illustration on the front and on the back contain a definition, a first language translation, an exemplary sentence, and questions that would engage the students in discussion about the words.

### Graphic organizers and foldables

Graphic organizers are visual and graphic displays that present relationships between facts, terms, and or ideas within a learning task. Types of graphic organizers include knowledge maps, concept maps, story maps, cognitive organizers, advance organizers, and concept diagrams. Graphic organizers support ELLs because they provide a means of displaying complex text succinctly and graphically.

### Introduction of key vocabulary; graphic organizers and foldables

Essential vocabulary should be provided as background information to help students focus on the increasingly complex mathematics in this lesson.

Although it is possible students may be familiar with some or all of these terms, it would be useful to provide an introduction or review. These six terms could be put into a foldable like this:

<http://www.lauracandler.com/filecabinet/literacy/PDFRead/VocabFoldDir.pdf>

For each term, have students write it accurately into their graphic organizer (foldable); provide an accurate auditory imprint by carefully pronouncing the word several times and having the class chorally repeat. Have students copy the concise definitions, and when applicable, add a sketch (such as for positive exponent) or example.

- **Positive Exponent:** (Noun.) Tells how many times to use the base number as a factor. “We read this [indicate  $4^8$ ] as four to the eighth power, or four to the power of eight. In this example, eight is the positive exponent. It tells us to multiply four times itself eight times. We can write this as  $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ .”
- **Negative Exponent:** (Noun.) Tells how many times to divide 1 by the number. “We read this [indicate  $5^{-3}$ ] as five to the negative three, or five to the power of negative three. In this example, negative three is a negative exponent. It tells us to divide 1 by 5, three times. We can write this as  $(1 / 5 \cdot 5 \cdot 5)$ .”
- **Integer:** (Noun.) All positive and negative counting numbers. “Four and negative four are examples of integers. Fractions like three eighths or decimals like negative seven tenths are not integers.”

- **Proof:** (Noun.) A series of steps to show something is true or accurate. Related to the verb prove. “Dania’s proof has four steps to show that  $x = y$  in this problem.”
- **Prove:** (Verb.) To show that something is true or accurate. Related to the noun proof. “We can prove that  $y = x$  by using the commutative property.”
- **Case:** (Noun.) An example or possibility. “In the first case, the answer is seven.”

## Common Core Inc. Socratic Discussion

The goal of this lesson is to show why the laws of exponents, (10)–(12) are correct for all integers  $a$  and  $b$  and for all  $x, y > 0$ . We recall (10)–(12):

For all  $x, y > 0$  and for all integers  $a$  and  $b$ , we have:

$$x^a \cdot x^b = x^{a+b} \quad (10)$$

$$(x^b)^a = x^{ab} \quad (11)$$

$$(xy)^a = x^a y^a \quad (12)$$

MP.  
7  
&  
...

This is a tedious process as the proofs for all three are somewhat similar. The proof of (10) is the most complicated of the three, but if one understands the proof of the easier identity (11), one will get a good idea how all three proofs go. Therefore, we will only prove (11) completely.

We have to first decide on a strategy to prove (11). Ask students what we already know about (11).

Elicit the following from the students:

- Equation (7) of Lesson 5 says for any positive  $x$ ,  $(x^m)^n = x^{mn}$  for all whole numbers  $m$  and  $n$ .

How does this help us? It tells us that:

(A) (11) is already known to be true when the integers  $a$  and  $b$ , in addition, satisfy  $a \geq 0, b \geq 0$ .

- Equation (9) of Lesson 5 says that the following holds:

(B)  $x^{-m} = \frac{1}{x^m}$  for any whole number  $m$ .

How does this help us? As we shall see from an exercise below, (B) is the statement that another special case of (11) is known.

- We also know that if  $x$  is positive, then

(C)  $\left(\frac{1}{x}\right)^m = \frac{1}{x^m}$  for any whole number  $m$ .

This is because if  $m$  is a positive integer, (C) is implied by equation (5) of Lesson 4, and if  $m = 0$ , then both sides of (C) are equal to 1.

How does this help us? We will see from another exercise below that (C) is in fact another special case of (11), which is already known to be true.

*Scaffolding:*

- Keep statements (A), (B), and (C) visible throughout lesson for reference purposes.

MP.

MP.



**The Laws of Exponents**

For  $x, y > 0$ , and all integers  $a, b$ , the following holds:

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a$$

Facts we will use to prove (11):

) (11) is already known to be true when the integers  $a$  and  $b$  satisfy  $a \geq 0, b \geq 0$ .

)  $x^{-m} = \frac{1}{x^m}$  for any whole number  $m$

)  $\left(\frac{1}{x}\right)^m = \frac{1}{x^m}$  for any whole number  $m$

**AIR New Activities****Background knowledge for teachers**

Relying on the “initiate, respond, evaluate” form of interaction (wherein the teacher poses a question, the student responds, and the teacher evaluates the response) is not conducive for ELLs to engage in meaningful mathematical discourse on this topic. Students will need the time to become familiar with the text before being asked to discuss it in class with peers.

**Clarification of key concepts**

By attending to the language load in the texts shared with students and in the explanations provided orally, key content can be highlighted and amplified. Some ways to clarify key concepts are using present tense, using shorter sentences, using examples related to school contexts, using graphics and arrows to illustrate points, and using white space and color to accentuate important information.

**Structured opportunities to speak with a partner or small group**

ELLs should be paired with more proficient English speakers for structured opportunities to practice speaking in English. In addition, there should be some initial training to assist pairs in working together. There are several excellent resources that provide guidance related to academic conversations and activities to help students acquire the skills they need to engage in these conversations. Examples of skills include elaborate and justify, support ideas with examples, build on or challenge a partner’s ideas, paraphrase, and synthesize conversation points. To assist ELLs, we recommend providing students with frames for prompting the skill. For example, for the skill “elaborate and clarify,” frames for prompting the skill might be “I am a little confused about the part...”; “Can you tell me more about...?”; “What do you mean by...?”; “Can you give me an example?”; “Could you try to explain it again?” Frames for responding might be “In other words...”; “It’s similar to when...”

**AIR Routines for Teachers****Scaffold Socratic discussion**

Scaffold this Socratic discussion such that students will have access to the student-friendly source material in advance, and pair them to begin initial conversations about the content with a partner before

moving to a whole-class discussion.

**Previous day** [to prepare for the Socratic discussion]

T: Your homework for tonight will be to read this document [see the following] that we will be talking about tomorrow in class. When you read it, you also will be writing as a way to get ready for our conversation tomorrow. You will be writing **notes, quotes, questions**, and a **summary**. You can write everything here on this paper, in the margins.

Your **notes** can be anything you notice about the reading. For example, you might write “This part is weird,” or “I think this means something different in my language.”

Your **quotes** are any sentences or equations in the document that you will want to talk about with all of us tomorrow.

Your **questions** will be to help you understand the ideas more clearly. Think of questions that need more than a one-word answer.

Your **summary** will be two or three sentences to explain the big ideas from the text. Write it on the back.

By purposefully choosing wording and the organization of the way the content is presented, students can better attend to the key concepts without the need to simultaneously decode more complex grammatical structures.

### Homework text for students

*You will write notes, quotes, questions, and a short summary.*

Remember that:

For all  $x, y > 0$  and for all integers  $a$  and  $b$ :

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a$$

- In our last lesson, Equation 7 said:

For any positive  $x$ ,  $(x^m)^n = x^{mn}$ . This is true for all whole numbers  $m$  and  $n$ .

How does this help us? It tells us that we already know:

$$(x^b)^a = x^{ab} \text{ when the integers } a \text{ and } b \text{ are both greater than or equal to zero } (a \geq 0, b \geq 0).$$

- In our last lesson, Equation 9 said:

$$\text{For any whole number } m, x^{-m} = 1/x^m.$$

How does this help us? Tomorrow we will talk about how  $x^{-m} = 1/x^m$  shows us that there is another special case of  $(x^b)^a = x^{ab}$ .

- We also know that:

$$\text{For any positive } x, (1/x)^m = 1/x^m \text{ for any whole number } m.$$

Why? Because if  $m$  is a positive integer,  $(1/x)^m = 1/x^m$  (This is what we learned in equation 5 of lesson 4.)

Also, if  $m = 0$ , then both sides of  $(1/x)^m = 1/x^m$  are equal to 1.

How does this help us? Tomorrow we will talk about how  $(1/x)^m = 1/x^m$  is another special case of  $(x^b)^a = x^{ab}$  (which is already known is true).

**AIR Additional Supports****Structured opportunities to speak with a partner or small group**

ELLs need multiple opportunities to try out and rehearse their new vocabulary and grammatical structures, and providing structured opportunities for students to communicate their thinking to one another is a low-stress way to facilitate this.

**Modeling and explanation**

Establish norms for the discussion so students know what is expected of them (see the table that follows). Arrange chairs in a circle or some other arrangement so that all students can see one another. Students should address one another and not the teacher. Allow students several minutes to compare assignments with one or two purposefully selected classmates, ensuring that each pair or group has students at different levels of English language proficiency and familiarity with the content. Their conversation should focus on their notes, quotes, questions, and summaries.

Before beginning the Socratic discussion, review the Socratic Discussion Overview of the process and expectations for communication. Post examples of the sentence frames in easily visible locations or provide a printed handout to each student.

<b>Socratic Discussion Overview</b> (adapted from Thompson & Radosavljevic, 2013)		
<b>Intention</b>	<b>Question</b>	<b>Examples</b>
Students engage in dialogue, not debate.	How can you politely address ideas you might not agree with?	“I heard you say _____. However, in my experience _____.” “How did you arrive at that view?”
Ensure equitable participation.	How can the discussion group monitor dominant participants and engage shy participants?	“I would like to hear what someone from this side of the group thinks about _____.” “Alex, what are your thoughts on what Marcela just said?”
Ask open-ended questions.	What is an open-ended question?	“How are these ideas different from the examples we used yesterday?”
Build on each other’s ideas.	How do you build on what someone else said?	“My idea is similar to Doug’s, but I think that....”
Refer to the text as often as possible.	What can you do when the conversation gets off topic?	“Let’s look at the reading again. I chose a quote from bullet 2. It says....”

Engage in Socratic discussion (focusing on notes, quotes, questions, and summaries) for several minutes to review source material provided (above) as homework.

### Common Core Inc. Exercises 1–3

Students complete Exercises 1–3 in small groups.

#### Exercise 1

Show that (C) is implied by equation (5) of Lesson 4 when  $m > 0$ , and explain why (C) continues to hold even when  $m = 0$ .

Equation (5) says, for any numbers  $x, y$ , ( $y \neq 0$ ) and any positive integer  $n$ , the following holds:  $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ . So,

$$\begin{aligned} \left(\frac{1}{x}\right)^m &= \frac{1^m}{x^m} && \text{By } \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \text{ for positive integer } n \text{ and nonzero } y \text{ (5)} \\ &= \frac{1}{x^m} && \text{Because } 1^m \end{aligned}$$

If  $m = 0$ , then the left side is

$$\begin{aligned} \left(\frac{1}{x}\right)^m &= \left(\frac{1}{x}\right)^0 \\ &= 1 && \text{By definition of } x^0 \end{aligned}$$

and the right side is

$$\begin{aligned} \frac{1}{x^m} &= \frac{1}{x^0} \\ &= \frac{1}{1} && \text{By definition of } x^0 \\ &= 1 \end{aligned}$$

#### AIR Additional Supports

##### Clarification of key concepts

ELLs at the entering, emerging, and transitioning levels should complete these exercises in purposefully selected pairs of students. The wording of the exercises should be clarified (see the following example) to highlight the key ideas, and the general outline or format for the proof should be provided (in the form of a T-chart that students should have prior experience in using for proofs).

##### AIR Routine for Teachers

##### Clarified wording

Exercise 1.

Show that for any positive  $x$ ,  $(1/x)^m = 1/x^m$  is true **when  $m = 0$** .

Provide students with the “statements and reasons” table that follows to organize their thinking.

Statements	Reasons
Given _____	Given _____
If $m = 0$ , the left side of the equation is _____	Because _____ (or By definition of _____)
If $m = 0$ , the right side of the equation is _____	Because _____ (or By definition of _____)
Therefore, _____.	

**Exercise 2**

Show that (B) is in fact a special case of (11) by rewriting it as  $(x^m)^{-1} = x^{(-1)m}$  for any whole number  $m$ , so that if  $b = m$  (where  $m$  is a whole number) and  $a = -1$ , (11) becomes (B).

$$(B) \text{ says } x^{-m} = \frac{1}{x^m}.$$

The left side of (B),  $x^{-m}$  is equal to  $x^{(-1)m}$ .

The right side of (B),  $\frac{1}{x^m}$ , is equal to  $(x^m)^{-1}$  by the definition of  $(x^m)^{-1}$  in Lesson 5.

Therefore, (B) says exactly that  $(x^m)^{-1} = x^{(-1)m}$ .

**Exercise 3**

Show that (C) is a special case of (11) by rewriting (C) as  $(x^{-1})^m = x^{m(-1)}$  for any whole number  $m$ . Thus, (C) is the special case of (11) when  $b = -1$  and  $a = m$ , where  $m$  is a whole number.

$$(C) \text{ says } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m.$$

The left side of (C) is equal to

$$\left(\frac{1}{x}\right)^m = (x^{-1})^m \quad \text{By definition of } x^{-1},$$

and right side of (C) is equal to

$$\frac{1}{x^m} = x^{-m} \quad \text{By definition of } x^{-m},$$

and the latter is equal to  $x^{m(-1)}$ . Therefore, (C) says  $(x^{-1})^m = x^{m(-1)}$  for any whole number  $m$ .

**Common Core Inc. Socratic Discussion****MP.**

In view of the fact that the reasoning behind the proof of (A) (Lesson 4) clearly cannot be extended to the case when  $a$  and/or  $b$  is negative, it may be time to consider proving (11) in several separate cases so that, at the end, these cases *together* cover all possibilities. (A) suggests that we consider the following four separate cases of identity (11):

$$(i) \ a, b \geq 0$$

$$(ii) \ a \geq 0, b < 0$$

$$(iii) \ a < 0 \text{ and } b \geq 0$$

*Scaffolding:*

- Have students think about the four quadrants of the plane.
- Read aloud the meaning of the four cases as you write them

(iv)  $a, b < 0$

- Ask students why there are no other possibilities.
- Ask students if we need to prove case (i).
- No, because (A) corresponds to case (i) of (11).

We will prove the three remaining cases in succession.

<b>AIR Additional Supports</b>
<p><b>Background knowledge for teachers, key academic vocabulary, and clarification of key concepts</b></p> <p>Because students will not have seen this source material in advance, students at the entering, emerging, and transitioning levels may be unprepared to participate in a genuine Socratic discussion of this content. This activity is best reworked by having students focus on the meanings of the different cases in pairs and then engage in a whole-group discussion and explanation.</p>
<b>AIR Routine for Teachers</b>
<p>With an assigned partner, students identify <math>a</math> and <math>b</math> on the coordinate plane for each case (i) – (iv). Refer back to vocabulary taught at the beginning of the lesson, and focus on the word <i>case</i>.</p> <p><b>T:</b> Let’s think about the different possibilities or <b>cases</b> for <math>a</math> and <math>b</math> being positive [teacher writes a positive symbol], negative [teacher writes a negative symbol], or equal to zero [teacher writes =0]. What are the different combinations we might have? Here is the first possibility:</p> <p>(i) <math>a, b \geq 0</math></p> <p><b>T:</b> In this example, both <math>a</math> and <math>b</math> are greater than or equal to zero. Both are positive.</p> <p><b>T:</b> Where would this be on the coordinate plane? Sketch a coordinate plane and write this in Quadrant I. Show your partner your work, and look at your partner’s work to make sure it is Quadrant I.</p>
<p><b>Teacher Modeling</b></p> <p>ELLs may not be clear about what instructions may mean, even though they may understand the concepts and content. Modeling what is expected for students is a way to ensure that students are clear about what the expectations for a particular part of the lesson may be.</p>
<b>AIR Routine for Teachers</b>
<p><b>T:</b> What is another possible combination for <math>a</math> and <math>b</math> being positive, negative, or equal to zero? Talk with your partner. Which quadrant would this belong in? Write it onto your coordinate plane. [Teacher models as students record in their notes.] Continue for remaining possibilities (cases).</p> <p><b>T:</b> Let’s label our cases to keep them organized.</p> <p>(ii) <math>a \geq 0, b &lt; 0</math></p> <p>(iii) <math>a &lt; 0</math> and <math>b \geq 0</math></p> <p>(iv) <math>a, b &lt; 0</math></p> <p><b>T:</b> So we see that we have four cases, and each one focuses on the values in different quadrants of the coordinate plane. Four cases, four quadrants.</p> <p><b>T:</b> Now let’s look again at <math>(x^b) = x^{ab}</math>. We know this is true for our first case (<math>a, b \geq 0</math>). Now we</p>

*Scaffolding:*

- Keep the example done previously with concrete numbers visible so students can relate the symbolic argument to the work just completed.
- Remind students that when concrete numbers are used, we can push through computations and show that the left side and right side are the same. For symbolic arguments we must look at each side

are going to see if this is true for each of the other cases as well.

### Common Core Inc. Socratic Discussion

**Case (ii):** We have to prove that for any positive  $x$ ,  $(x^b)^a = x^{ab}$ , when the integers  $a$  and  $b$  satisfy  $a \geq 0, b < 0$ . For example, we have to show that  $(5^{-3})^4 = 5^{(-3)4}$ , i.e.,  $(5^{-3})^4 = 5^{-12}$ . The following is the proof:

$$\begin{aligned}
 (5^{-3})^4 &= \left(\frac{1}{5^3}\right)^4 && \text{By definition} \\
 &= \frac{1}{(5^3)^4} && \text{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m \text{ (C)} \\
 &= \frac{1}{5^{12}} && \text{By } (x^m)^n = x^{mn} \text{ for all whole numbers } m \text{ and } n \text{ (A)} \\
 &= 5^{-12} && \text{By definition}
 \end{aligned}$$

In general, we just imitate this argument. Let  $b = -c$ , where  $c$  is a positive integer. We now show that the left side and the right side of  $(x^b)^a = x^{ab}$  are equal. The left side is:

$$\begin{aligned}
 (x^b)^a &= (x^{-c})^a \\
 &= \left(\frac{1}{x^c}\right)^a && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \\
 &= \frac{1}{(x^c)^a} && \text{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m \text{ (C)} \\
 &= \frac{1}{x^{ac}} && \text{By } (x^m)^n = x^{mn} \text{ for all whole numbers } m \text{ and } n \text{ (A)}
 \end{aligned}$$

The right side is:

$$\begin{aligned}
 x^{ab} &= x^{a(-c)} \\
 &= x^{-(ac)} \\
 &= \frac{1}{x^{ac}} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}
 \end{aligned}$$

The left side and the right side are equal, thus, *case (ii)* is done.

**Case (iii):** We have to prove that for any positive  $x$ ,  $(x^b)^a = x^{ab}$ , when the integers  $a$  and  $b$  satisfy  $a < 0$  and  $b \geq 0$ . This is very similar to *case (ii)*, so it will be left as an exercise.

Students complete Exercise 4 independently or in pairs.

#### Exercise 4

*Proof of Case (iii):* Show that when  $a < 0$  and  $b \geq 0$ ,  $(x^b)^a = x^{ab}$  is still valid. Let  $a = -c$  for some positive integer  $c$ . Show that the left side and right sides of  $(x^b)^a = x^{ab}$  are equal.

The left side is

$$\begin{aligned} (x^b)^a &= (x^b)^{-c} \\ &= \frac{1}{(x^b)^c} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \\ &= \frac{1}{x^{cb}} && \text{By } (x^m)^n = x^{mn} \text{ for all whole numbers } m \text{ and } n \text{ (A)} \end{aligned}$$

The right side is

$$\begin{aligned} x^{ab} &= x^{(-c)b} \\ &= x^{-(cb)} \\ &= \frac{1}{x^{cb}} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \end{aligned}$$

So the two sides are equal.

#### AIR Additional Supports

##### Background knowledge for teachers

To make this a more meaningful, memorable, and engaging portion of the lesson for ELLs, the proof should be organized into a T-chart and students should be provided with the opportunity to turn-and-talk at various points during the explanation. Students also will compare and contrast cases *ii* and *iii* using a Venn diagram to highlight similarities and differences.

##### Clarification of key concepts and key academic vocabulary

By attending to the language load in the texts shared with students and in the explanations provided orally, key content can be highlighted and amplified. Some ways to clarify key concepts include using present tense, using shorter sentences, using examples related to school contexts, using graphics and arrows to illustrate points, and using white space and color to accentuate important information.

#### AIR Routine for Teachers

##### Structured opportunities to speak with a partner or small group

**T:** To make this clearer, we will substitute integers into this equation. I will write it as a proof, in two columns.

After writing the first two lines of the proof (the given or “by definition,” and then the rewriting of the first fractional term), have students work with a partner to simplify it further, down to the final solution of  $5^{-12}$ . Explain each step of the reasons, and explain that the term *by* is used to mean the same thing as *because* when referring back to earlier work or already proven examples.



**T:** Now that we have created the proof of this case with integers, you will work with your partner to create the proof of this case with variables. Your work will follow the exact same format.

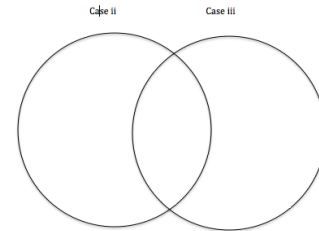
Have students work with partners to redraft the same proof but with variables instead of integers.

Explore case iii (Exercise 4) in the same fashion, first with an integer substitute, and then with variables. Ensure that students have opportunities to speak with their partners.

After completion of cases ii and iii:

*How are cases ii and iii similar and how are they different?*

Provide a Venn diagram (or have students draw it) and with partners, generate a few key ideas for each category.



### Common Core Inc. Socratic Discussion

The only case remaining in the proof of (11) is *case (iv)*. Thus we have to prove that for any positive  $x$ ,  $(x^b)^a = x^{ab}$  when the integers  $a$  and  $b$  satisfy  $a < 0$  and  $b < 0$ . For example,  $(7^{-5})^{-8} = 7^{5 \cdot 8}$  because

$$\begin{aligned} (7^{-5})^{-8} &= \frac{1}{(7^{-5})^8} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \\ &= \frac{1}{7^{-(5 \cdot 8)}} && \text{By case (ii)} \\ &= 7^{5 \cdot 8} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \end{aligned}$$

In general, we can imitate this explicit argument with numbers as we did in case (ii). Let  $a = -c$  and  $b = -d$ , where  $c$  and  $d$  are positive integers. Then, the left side is:

$$\begin{aligned} (x^b)^a &= (x^{-c})^{-d} \\ &= \frac{1}{(x^{-c})^d} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \\ &= \frac{1}{x^{-cd}} && \text{By case (ii)} \\ &= \frac{1}{x^{cd}} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)} \\ &= x^{cd} && \text{By invert-and-multiply for division of complex fractions} \end{aligned}$$

The right side is:

$$\begin{aligned} x^{ab} &= x^{(-c)(-d)} \\ &= x^{cd} \end{aligned}$$

The left side is equal to the right side, thus, *case (iv)* is finished. Putting all of the cases together, the proof of (11) is complete. We now know that (11) is true for any positive integer  $x$  and any integers  $a$ ,  $b$ .

#### Scaffolding:

- Students may ask why the exponent  $c$  remained negative in the second line while  $d$  became positive. Reconcile this through the use of a concrete example or by pointing to the previous problem
 
$$(7^{-5})^{-8} = \frac{1}{(7^{-5})^8}$$
 first line.

### AIR Additional Supports

**Clarification of key concepts**

Following the protocol established in examining cases ii and iii, the creation of a pair of T-chart proofs is useful here. To further enrich the experience of the students at all levels of English proficiency, they should be invited to make conjectures about what will happen.

**T:** Look at case  $\underline{vi}$  ( $a, b < 0$ ). Find it on your coordinate plane. Which quadrant is it in?

**AIR Routine for Teachers****Structured opportunities to speak with a partner or small group**

**T:** Based on what we just did with the last cases, what do you think we should do with this case?  
Turn and talk with your neighbor.

Students will mention substitution of integers for variables and using a T-chart to organize the proofs.

**T:** Work with your partner to complete as much of this proof as you can.

After students have had some time to work, match pairs of students together.

**T:** Now you and your partner are going to get together with another pair, to make a group of four students. Share what you have been working on, and listen carefully as your new group members explain their thinking.

Then review as a whole class to ensure that students have successfully completed the proof.

**Background knowledge for teachers**

ELLs may not be familiar with the phrase “invert and multiply for division of complex fractions” as seen in the lesson plan that follows and in the completed proof.

**Common Core Inc. Closing**

Summarize, or have students summarize, the lesson.

Students have proven that the Laws of Exponents are valid for any integer exponent

**Common Core Inc. Exit Ticket**

9. Show directly that for any positive integer  $x$ ,  $x^{-5} \cdot x^{-7} = x^{-12}$ .

10. Show directly that for any positive integer  $x$ ,  $(x^{-2})^{-3} = x^6$ .

**AIR Additional Supports****Clarification of key concepts and use of a graphic organizer or foldable to organize ideas**

Have students complete a graphic organizer (below) with a partner to help them synthesize ideas and write independently.

With a partner, complete the following 3-2-1 summary graphic organizer. Use the word bank.

3 details from today’s lesson (write or draw):

--	--	--

2 big ideas from today’s lesson (write or draw):

1 summary sentence to explain what you learned.		
Today I learned that _____		
Word Bank		
proof	prove	positive exponent
negative exponent	case	integers

### Common Core Inc. Problem Set Sample Solutions

1. You sent a photo of you and your family on vacation to seven Facebook friends. If each of them sends it to five of their friends, and each of those friends sends it to five of their friends, and those friends send it to five more, how many people (not counting yourself) will see your photo? No friend received the photo twice. Express your answer in exponential notation.

<i># of New People to View Your Photo</i>	<i>Total # of People to View Your Photo</i>
<b>7</b>	<b>7</b>
<b><math>5 \times 7</math></b>	<b><math>7 + (5 \times 7)</math></b>
<b><math>5 \times 5 \times 7</math></b>	<b><math>7 + (5 \times 7) + (5^2 \times 7)</math></b>
<b><math>5 \times 5 \times 5 \times 7</math></b>	<b><math>7 + (5 \times 7) + (5^2 \times 7) + (5^3 \times 7)</math></b>

The total number of people who viewed the photo is  $(5^0 + 5^1 + 5^2 + 5^3) \times 7$ .

2. Show directly, without using (11), that  $(1.27^{-36})^{85} = 1.27^{-36 \cdot 85}$ .

$$\begin{aligned}
 (1.27^{-36})^{85} &= \left(\frac{1}{1.27^{36}}\right)^{85} && \text{By definition.} \\
 &= \frac{1}{(1.27^{36})^{85}} && \text{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m \text{ (C)} \\
 &= \frac{1}{1.27^{36 \cdot 85}} && \text{By } (x^m)^n = x^{mn} \text{ for whole numbers } m \text{ and } n \text{ (7)} \\
 &= 1.27^{-36 \cdot 85} && \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}
 \end{aligned}$$

3. Show directly that  $\left(\frac{2}{13}\right)^{-127} \cdot \left(\frac{2}{13}\right)^{-56} = \left(\frac{2}{13}\right)^{-183}$ .

$$\begin{aligned}
 \left(\frac{2}{13}\right)^{-127} \cdot \left(\frac{2}{13}\right)^{-56} &= \frac{1}{\left(\frac{2}{13}\right)^{127}} \cdot \frac{1}{\left(\frac{2}{13}\right)^{56}} && \text{By definition} \\
 &= \frac{1}{\left(\frac{2}{13}\right)^{127} \cdot \left(\frac{2}{13}\right)^{56}} && \text{By the Product Formula for Complex Fractions}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\left(\frac{2}{13}\right)^{127+56}} \quad \text{By } \mathbf{x^m \cdot x^n = x^{m+n}} \text{ for whole numbers } \mathbf{m} \text{ and } \mathbf{n} \text{ (6)} \\
 &= \frac{1}{\left(\frac{2}{13}\right)^{183}} \\
 &= \left(\frac{2}{13}\right)^{-183} \quad \text{By } \mathbf{x^{-m} = \frac{1}{x^m}} \text{ for any whole number } \mathbf{m} \text{ (B)}
 \end{aligned}$$

4. Prove for any positive number  $x$ ,  $x^{-127} \cdot x^{-56} = x^{-183}$

$$\begin{aligned}
 x^{-127} \cdot x^{-56} &= \frac{1}{x^{127}} \cdot \frac{1}{x^{56}} \quad \text{By definition} \\
 &= \frac{1}{x^{127 \cdot x^{56}}} \quad \text{By the Product Formula for Complex Fractions} \\
 &= \frac{1}{x^{127+56}} \quad \text{By } \mathbf{x^m \cdot x^n = x^{m+n}} \text{ for whole numbers } \mathbf{m} \text{ and } \mathbf{n} \text{ (6)} \\
 &= \frac{1}{x^{183}} \\
 &= x^{-183} \quad \text{By } \mathbf{x^{-m} = \frac{1}{x^m}} \text{ for any whole number } \mathbf{m} \text{ (B)}
 \end{aligned}$$

5. Prove for any positive number  $x$ ,  $x^{-m} \cdot x^{-n} = x^{-m-n}$  for positive integers  $m$  and  $n$ .

$$\begin{aligned}
 x^{-m} \cdot x^{-n} &= \frac{1}{x^m} \cdot \frac{1}{x^n} \quad \text{By definition} \\
 &= \frac{1}{x^m \cdot x^n} \quad \text{By the Product Formula for Complex Fractions} \\
 &= \frac{1}{x^{m+n}} \quad \text{By } \mathbf{x^m \cdot x^n = x^{m+n}} \text{ for whole numbers } \mathbf{m} \text{ and } \mathbf{n} \text{ (6)} \\
 &= x^{-(m+n)} \quad \text{By } \mathbf{x^{-m} = \frac{1}{x^m}} \text{ for any whole number } \mathbf{m} \text{ (B)} \\
 &= x^{-m-n}
 \end{aligned}$$

6. Which of the preceding four problems did you find easiest to do? Explain.

*Students will likely say that  $x^{-m} \cdot x^{-n} = x^{-m-n}$  (Problem 5) was the easiest problem to do. It requires the least amount of writing because the symbols are easier to write than decimal or fraction numbers.*

7. Use the properties of exponents to write an equivalent expression that is a product of distinct primes, each raised to an integer power.

$$\frac{10^5 \cdot 9^2}{6^4} = \frac{(2 \cdot 5)^5 \cdot (3 \cdot 3)^2}{(2 \cdot 3)^4} = \frac{2^5 \cdot 5^5 \cdot 3^2 \cdot 3^2}{2^4 \cdot 3^4} = 2^{5-4} \cdot 3^{4-4} \cdot 5^5 = 2^1 \cdot 3^0 \cdot 5^5 = 2^1 \cdot 1 \cdot 5^5 = 2 \cdot 5^5$$

**AIR Additional Supports****Clarification of key concepts**

For the problem set, ELLs may benefit from clarification of key concepts to make the meaning of the problems more comprehensible and streamlined. This may include ensuring that the context is familiar for all students and school-centered, using present tense, and breaking lengthy sentences into shorter sentences. For example, substitute this text for problem 1.

Tell seven friends a funny joke. Each friend tells your joke to five of *their* friends. Then each of those five friends tells the joke to five more people. No one heard the joke more than one time.

How many people (not including you) will hear the joke? Express your answer in exponential notation.

## Algebra I, Module 3, Lesson 5: The Power of Exponential Growth

### Overview

The following table outlines the scaffolds that have been added to support ELLs throughout the Common Core Inc. Lesson 5, The Power of Exponential Growth.

*AIR New Activity* refers to an activity not in the original lesson that AIR has inserted into the original lesson. *AIR Additional Supports* refer to additional supports added to a component already in place in the original lesson. *AIR new activities* and *AIR additional supports* are boxed whereas the text that is in the original lesson is generally not boxed. *AIR Routines for Teachers* are activities that include instructional conversations that take place between teachers and students. In the *AIR Routines for Teachers* the text of the original Common Core Inc. lessons appears in standard black, whereas the AIR additions to the lessons are in green.

Original Component by Common Core, Inc.	AIR Additional Supports	AIR New Activities Type of Scaffold
<i>None included</i>		Cueing
<i>None included</i>		Key academic vocabulary Graphic organizers or foldables
Opening exercise	Clarification of key concepts	
Example 1	Clarification of key concepts Background knowledge for students Key academic vocabulary	
Discussion	Writing Structured opportunities for students to speak with a partner or small group	
Example 2	<i>No changes recommended</i>	
Exercise 1	Use of multimedia to enhance comprehension Background knowledge for teachers	
Exercise 2	Clarification of key concepts Background knowledge for students	
Closing	Teacher modeling and explanation Structured opportunities for students to speak with a partner or small group	
Exit ticket	Clarification of key concepts Graphic organizers and foldables	
Problem set	Graphic organizers and foldables Concrete and visual models Clarify key concepts	

## Common Core Inc. Lesson Introduction

### Student Outcomes

- Students are able to model with and solve problems involving exponential formulas.

### Lesson Notes

The primary goals of the lesson are to explore the connection between exponential growth and geometric sequences and to compare linear growth to exponential growth in context. In one exercise, students graph both types of sequences on one graph to help visualize the growth in comparison to each other. In the Closing, students are challenged to recognize that depending on the value of the base in the exponential expression of a geometric sequence, it can take some time before the geometric sequence will exceed the arithmetic sequence. Students begin to make connections in this lesson between geometric sequences and exponential growth and between arithmetic sequences and linear growth. These connections are formalized in later lessons.

#### AIR New Activity

##### Cueing: introduce objectives, student outcomes and key vocabulary for lesson

As a way to introduce students to the content, vocabulary, and agenda for the day, cueing can be very useful. Providing students with an orientation to the day's work is valuable and can help ELLs know which parts of the day's activities are most essential. The introduction of key vocabulary will be beneficial because students can attend to uses of the terms throughout the lesson.

#### AIR Routine for Teachers

Write on board and read aloud: *I will model and solve problems using exponential formulas.*

**T:** In this lesson, you will practice modeling and solving problems using exponential formulas. Let's review these three important vocabulary words: model, exponential, and formula.

Write on board and read aloud:

**T:** Model (noun): An example that is useful for others to copy.

Model (verb): To demonstrate, draw, or write down a good example for how something is done.

**T:** Use this word in a sentence as a noun. For example, look at where I have written the date on the board. This is a model for how you can write the date. It is a useful example for you to copy. Allow students time to turn to a neighbor and try using the word.

**T:** Now, use this word in a sentence as a verb. For example, I will model how to put the lid on this marker. I am demonstrating a good example for how this is done. I am modeling putting the lid on the marker. Now, you think of an example. Allow students time to turn to a neighbor and try using the word as a verb.

**T:** Our next word is exponential. What root word do you notice? Turn and tell your neighbor. Now let's say the root word together: exponent. (Underline the word exponent in the day's objective.) Remember that the exponent of a number tells us how many times to use that number in multiplication. So if I write  $4^5$ , it means I multiply 4 times itself five times. Tell

your partner an example of an exponent and what it means.

**T:** The word exponent is a noun, but the word exponential is an adjective, so we use it to describe other nouns. It means including or using exponents.

**T:** Our last word for now is formula. This is a noun that you have seen in earlier lessons. It has lots of meanings that you might know from science class or from home, but in mathematics class, a formula is a fact or rule that we show using letters and numbers. For example, the formula for finding the area of a rectangle is length times width, which we write as  $l \times w$ . Today, we will be working with exponential formulas, or rules or facts that we show using letters and numbers that include exponents.

Introduce the day's agenda by posting it and reviewing it orally while indicating each activity for the day's lesson.

### AIR New Activity

#### Key academic vocabulary, structured opportunities to speak with a partner or small group, background knowledge for students, and concrete and visual models

Providing opportunities for students to practice using their new vocabulary is essential, and speaking with a partner or small group in a structured way may be less stressful for students than speaking to the whole group. Using graphic organizers or foldables can help students further reinforce their retention of terms and definitions, and these graphic organizers also can serve as study guides and review materials.

### AIR Routine for Teachers

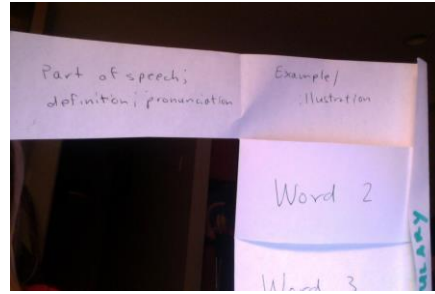
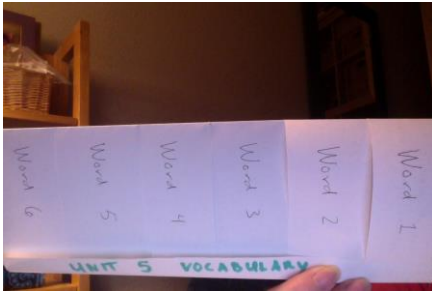
#### Foldable

Students will use a foldable (see the photographs that follow) to record vocabulary information for this lesson. This foldable is in a style called a “matchbook” style. Although the example in the illustration has space for six terms, it can be adjusted for greater or fewer terms; it depends on the unit. The key academic words or terms are written on the outside on the flaps, and the inside of the flaps list the part of speech, the definition, and pronunciation. The opposite side includes an example or an illustration, if applicable.

**T:** We are going to review some more of the important terms we will need for this lesson. You will record these in today's foldable for our vocabulary. The first word is an adjective, arithmetic. It LOOKS like the noun, arithmetic, which is a fancy word for mathematics, but it SOUNDS different, because it is pronounced air-ith-Mæt-ic. Say it with me: arithmetic (air-ith-Mæt-ic). This is an adjective and it describes a kind of pattern of numbers. An arithmetic sequence or pattern is made by adding the same value each time. An example is 2, 4, 6, 8, because we add 2 each time. What would be next in this arithmetic sequence?

**T:** The next word is geometric. This is an adjective, just like arithmetic, and is used to describe patterns and sequences. A geometric sequence is made by multiplying by the same value each time. An example is 75, 15, 3, 3/5, because we multiply by 1/5 each time. What would be next in this geometric sequence?





### Patterns and Sequence Sort

Have the students complete a sort activity. This formative assessment will provide the teacher with background information from a previous lesson on arithmetic and geometric sequences and also will provide students an opportunity to review these concepts.

#### Directions

- Enlarge, copy on cardstock, laminate, and cut out all the cards (below) for each pair or group.
- Give one set to each group and have them sort the cards into three groups (Arithmetic Sequence, Geometric Sequence, or Neither).
- The teacher will walk around the room and monitor the groups.
- Optional, after cards are sorted:
  - Determine the common difference or ratio.
  - Determine the seventh term of the sequence.

Arithmetic Sequence	Geometric Sequence	Neither
3.5, 4.3, 5.1, 5.9...	5, 15, 45, 135...	45, 43, 39, 33...
100, 75, 50, 25...	$\frac{1}{4}, \frac{1}{2}, 1, 2...$	-6, -5, -2, 3...
23, 18, 13, 8...	125, 75, 45, 27...	7, 8, 9, 11, 13...
29, 40, 51, 62...	$\frac{18}{25}, \frac{3}{5}, \frac{1}{2}, \frac{5}{12}...$	-14, -7, -2, -1, 4...
-2, -5, -8, -11...	1000, 100, 10, 1...	8, 12, 18, 27...
-10, -3, 4, 11...	$5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}...$	-2, -2, -2, -2...

## Common Core Inc. Opening Exercise

Direct students to begin the lesson with the following comparison of two options.

Two equipment rental companies have different penalty policies for returning a piece of equipment late:

Company 1: On day 1, the penalty is \$5. On day 2, the penalty is \$10. On day 3, the penalty is \$15. On day 4, the penalty is \$20 and so on, increasing by \$5 each day the equipment is late.

Company 2: On day 1, the penalty is \$0.01. On day 2, the penalty is \$0.02. On day 3, the penalty is \$0.04. On day 4, the penalty is \$0.08 and so on, doubling in amount each additional day late.

Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger 15 days late, he was shocked by the penalty fee. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	Penalty
1	\$5
2	\$10
3	\$15
4	\$20
5	\$25
6	\$30
7	\$35
8	\$40
9	\$45
10	\$50
11	\$55
12	\$60
13	\$65
14	\$70
15	\$75

Company 2	
Day	Penalty
1	\$0.01
2	\$0.02
3	\$0.04
4	\$0.08
5	\$0.16
6	\$0.32
7	\$0.64
8	\$1.28
9	\$2.56
10	\$5.12
11	\$10.24
12	\$20.48
13	\$40.96
14	\$81.92
15	\$163.84

- Which company has a greater 15-day late charge?

*Company 2*

- Describe how the amount of the late charge changes from any given day to the next successive day in both Companies 1 and 2.

*For Company 1, the change from any given day to the next successive day is an increase by \$5. For Company 2, the change from any given day to the next successive day is an increase by a factor of 2.*

- How much would the late charge have been after 20 days under Company 2?

**\$5,242.88**

Then discuss the following:

- Write a formula for the sequence that models the data in the table for Company 1.
  - $f(n) = 5n$ , where  $n$  begins with 1.
- Is the sequence Arithmetic, Geometric, or neither?
  - Arithmetic
- Write a formula for the sequence that models the data in the table for Company 2.
  - $f(n) = 0.01(2)^{n-1}$ , where  $n$  begins with 1.
- Is the sequence Arithmetic, Geometric, or neither?
  - Geometric
- Which of the two companies would you say grows more quickly? Why?
  - The penalty in Company 2 grows more quickly after a certain time because each time you are multiplying by 2 instead of just adding 5.

### AIR Additional Support

#### Background knowledge for teachers

At times, it may be useful to have additional information about how students may have been educated outside the United States and to consider which contexts may be unfamiliar for students.

The opening exercise from the original lesson has been revised. Differentiating in this manner allows for students to access the mathematics using visual supports and reduced or simplified language modifications. In addition, through effective questioning and prompting, teachers can support students in both groups and then engage the whole class to discuss the problem.

#### Clarification of key concepts: alternate opening exercise to substitute for original opening exercise

By attending to the language load in the texts shared with students and in the explanations provided orally, key content can be highlighted and amplified. Some ways to clarify key concepts include using present tense, using shorter sentences, using examples related to school contexts, using graphics and arrows to illustrate main ideas, and using white space and color to accentuate important information.

#### AIR Routine for Teachers

##### Use this alternate exercise for ELLs who may benefit.

Jim rents a bike from Company 2. Jim thinks Company 2 will be a better value. Jim keeps the bike longer than expected. Jim returns the bike **15** days late.

- How much money does Jim pay to rent the bike from Company 2?
- How much money would Jim pay if he used Company 1 instead?

Record your answers by continuing the two tables below. Calculate for Day 5, Day 6, and so on, through Day 15.

Rental Company Late-Fee Policy			
Company 1		Company 2	
Day 1	\$5	Day 1	\$0.01
Day 2	\$10	Day 2	\$0.02
Day 3	\$15	Day 3	\$0.04
Day 4	\$20	Day 4	\$0.08
Day 5	?	Day 5	?
Day 6	?	Day 6	?
Day 7	?	Day 7	?
...	?	...	?
Day 15	?	Day 15	?

The fee increases \$5 each day the bike is late.      The fee doubles in amount each day the bike is late.

### AIR Additional Supports

#### Structured opportunities to speak with a partner or small group, graphic organizers and recording and processing new ideas

Have students record information for the first 15 days in two tables (as in the original problem). Working in groups, the students will work with a shoulder partner to discuss and solve the following problems. Be purposeful in pairing students together on the basis of language proficiency and mathematical ability. Students also may need additional sentence prompts to support their academic language.

#### Structured opportunities to speak with a partner or small group

To discuss the findings in small groups (four students) and to allow for greater participation of all students, print and cut out the cards that follow. Explain to the students how the activity will work.

Step 1. Student 1 holds question cards in a fan and says, “Pick a card, any card!”

Step 2. Student 2 picks a card, reads the question aloud, and allows think or wait time.

Step 3. Student 3 answers the question.

Step 4. Student 4 checks the answer and then restates the answer (or adds more details).

Step 5. Students rotate roles for each new round.

Which company has a greater 15-day late charge?	How does the late charge change from one day to the next for Company 1?	How does the late charge change from one day to the next for Company 2?
How much is the late fee after 20 days at Company 1?	Write a formula for the sequence that models the data in the table for Company 1.	Write a formula for the sequence that models the data in the table for Company 2.
How much is the late fee after 20 days at Company 2?	Is the sequence for Company 1 arithmetic, geometric, or neither?	Is the sequence for Company 2 arithmetic, geometric, or neither?

Whole class discussion. Briefly discuss as a whole class which of the two companies' fees would grow more quickly. Why?

The penalty in Company 2 grows more quickly after a certain time because each time you are multiplying by 2 instead of just adding 5.

## Common Core Inc. Example

### Example 1

Folklore suggests that when the creator of the game of chess showed his invention to the country's ruler, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice. The inventor, being rather clever, said he would take a grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four on the third square, eight on the fourth square, and so on, doubling the number of grains of rice for each successive square. The ruler was surprised, even a little offended, at such a modest prize, but he ordered his treasurer to count out the rice.

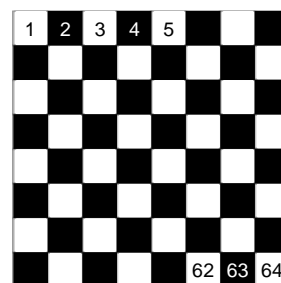
- a. Why is the ruler "surprised"? What makes him think the inventor requested a "modest prize"?

*The ruler is surprised because he hears a few grains mentioned—it seems very little, but he does not think through the effect of doubling each collection of grains; he does not know that the amount of needed rice will grow exponentially.*

The treasurer took more than a week to count the rice in the ruler's store, only to notify the ruler that it would take more rice than was available in the entire kingdom. Shortly thereafter, as the story goes, the inventor became the new king.

- b. Imagine the treasurer counting the needed rice for each of the **64** squares. We know that the first square is assigned a single grain of rice, and each successive square is double the number of grains of rice of the former square. The following table lists the first five assignments of grains of rice to squares on the board. How can we represent the grains of rice as exponential expressions?

Square #	Grains of Rice	Exponential Expression
1	1	$2^0$
2	2	$2^1$
3	4	$2^2$
4	8	$2^3$
5	16	$2^4$



- c. Write the exponential expression that describes how much rice is assigned to each of the last three squares of the board.

Square #	Exponential
62	$2^{61}$
63	$2^{62}$
64	$2^{63}$

### AIR Additional Supports

#### Clarification of key concepts, background knowledge for students, and key academic vocabulary

The language load in this example is quite heavy, using nonessential words in less common ways, like the term *ruler* being used to describe a leader, *store* being used to describe a warehouse, and low-frequency words like *modest*. If possible, students should preview the picture book, *One Grain of Rice: A Mathematical Folktale* (Demi, 1997) before this class meeting (perhaps in their English class) to be familiar with the story. This would give teachers the opportunity to support students' language development and background knowledge.

### AIR Routine for Teachers

Teach the meaning of the word *successive*. Students may add it to their foldable created earlier in the lesson. Explain that it is an adjective, and is in some ways related to the word *next*. It describes one thing following another. Even though it sounds like the word *success*, it has a different meaning in this context. *I rode my bicycle on 3 successive days, Tuesday, Wednesday, and Thursday.*

In addition, the text for this example may be presented to students as follows:

This is a folktale about the game of chess. A long time ago, when the inventor of the game of chess showed the game to the country's leader, the leader was very impressed. The leader told the inventor to think of a prize for his work. Look at the illustration of the chessboard. The inventor said he would take one grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four grains of rice on the third square, eight on the fourth square, and so on. The number of grains of rice doubles for each following square.

The country's leader was surprised, but he began to count out the rice.

- a. Why do you think the country's leader was surprised?

The country's leader began to count out the grains of rice for each square, but soon ran out. He would need more rice than was available in the entire country!

- b. The first square on the chessboard has one grain of rice, or  $2^0$ . The second square has two grains of rice, or  $2^1$ . The third square has four grains of rice, or  $2^2$ , and the fourth square has eight grains of rice, or  $2^3$ . Using this information, complete the table (following) to record the number of grains of rice in each square. How can we represent the grains of rice as exponential expressions?
- c. Write the exponential expression to describe how much rice is necessary for each of the last three squares of the board.

## Common Core Inc. Discussion/Writing Exponential Formulas

Ask students to consider how the exponential expressions of Example 1, part (b) relate to one another.

MP.

- Why is the base of the expression 2?
  - Since each successive square has twice the amount of rice as the former square, the factor by which the rice increases is a factor of 2.
- What is the explicit formula for the sequence that models the number of rice grains in each square? Use  $n$  to represent the number of the square and  $f(n)$  to represent the number of rice grains assigned to that square.
  - $f(n) = 2^{(n-1)}$ , where  $f(n)$  represents the number of rice grains belonging to each square, and  $n$  represents the number of the square on the board.
- Would the formula  $f(n) = 2^n$  work? Why or why not?
  - No, the formula is supposed to model the numbering scheme on the chessboard corresponding to the story.
- What would have to change for the formula  $f(n) = 2^n$  to be appropriate?
  - If the first square started with 2 grains of rice and doubled thereafter, or if we numbered the squares as starting with square number 0 and ending on square 63, then  $f(n) = 2^n$  would be appropriate.
- Suppose instead that the first square did not begin with a single grain of rice but with 5 grains of rice, and then the number of grains was doubled with each successive square. Write the sequence of numbers representing the number of grains of rice for the first five squares.
  - 5, 10, 20, 40, 80
- Suppose we wanted to represent these numbers using exponents? Would we still require the use of the powers of 2?
  - Yes.  $5 = 5(2^0)$ ,  $10 = 5(2^1)$ ,  $20 = 5(2^2)$ ,  $40 = 5(2^3)$ ,  $80 = 5(2^4)$
- Generalize the pattern of these exponential expressions into an explicit formula for the sequence. How does it compare to the formula in the case where we began with a single grain of rice in the first square?
  - $f(n) = 5(2^{n-1})$ , the powers of 2 cause the doubling effect, and the 5 represents the initial 5 grains of rice.
- Generalize the formula even further. Write a formula for a sequence that allows for any possible value for the number of grains of rice on the first square.
  - $f(n) = a2^{n-1}$ , where  $a$  represents the number of rice grains on the first square.
- Generalize the formula even further. What if instead of doubling the number of grains, we wanted to triple or quadruple them?
  - $f(n) = ab^{n-1}$ , where  $a$  represents the number of rice grains on the first square and  $b$  represents the factor by which the number of rice grains is multiplied on each successive square.
- Is the sequence for this formula Geometric, Arithmetic, or neither?

## — Geometric

**AIR Additional Support****Structured opportunities to speak with a partner or small group**

Remind students of the terms *exponent* and *base*. Review the word *successive* from earlier in the lesson. Provide a word bank and sentence starters (as in the following example) to scaffold participation of ELLs at the entering, emerging, and transitioning levels. Have students write their responses using the sentence starters and word bank before sharing with their partners.

Word Bank		
Nouns	Adjectives	Verbs
base exponent formula model pattern sequence	arithmetic exponential formulaic geometric successive	model

**Sentence starters**

*The base of the expression is 2 because \_\_\_\_\_.*

*I think the formula (would/would not) work because \_\_\_\_\_.*



## Common Core Inc. Example 2

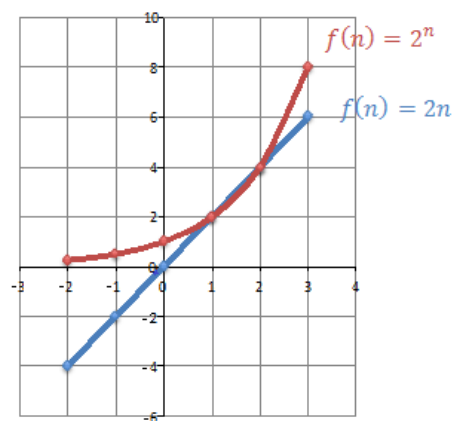
Note that students may, or may not, connect the points in their graphs with a smooth line or curve as shown below. Be clear with the students that we were only asked to graph the points but that it is natural to recognize the form that the points take on by modeling them with the line or curve.

**Example 2**

Let us understand the difference between  $f(n) = 2n$  and  $f(n) = 2^n$ .

- a. Complete the tables below, and then graph the points  $(n, f(n))$  on a coordinate plane for each of the formulas.

$n$	$f(n) = 2n$
-2	-4
-1	-2
0	0
1	2
2	4
3	6



$n$	$f(n) = 2^n$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

- b. Describe the change in each sequence when  $n$  increases by 1 unit for each sequence.

For the sequence  $f(n) = 2n$ , for every increase in  $n$  by 1 unit, the  $f(n)$  value increases by 2 units. For the sequence  $f(n) = 2^n$ , for every increase in  $n$  by 1 unit, the  $f(n)$  value increases by a factor of 2.

## Common Core Inc. Exercise 1

Students should attempt Exercises 1 and 2 independently and then share responses as a class.

### Exercise 1

A typical thickness of toilet paper is 0.001 inches. This seems pretty thin, right? Let's see what happens when we start folding toilet paper.

- a. How thick is the stack of toilet paper after **1** fold? After **2** folds? After **5** folds?

**After 1 fold:  $0.001(2^1) = 0.002$ "**

**After 2 folds:  $0.001(2^2) = 0.004$ "**

**After 5 folds:  $0.001(2^5) = 0.032$ "**

- b. Write an explicit formula for the sequence that models the thickness of the folded toilet paper after **n** folds.

**$f(n) = 0.001(2^n)$**

- c. After how many folds will the stack of folded toilet paper pass the **1** foot mark?  
After **14** folds.

- d. The moon is about **240,000** miles from Earth. Compare the thickness of the toilet paper folded **50** times to the distance from Earth.

*Toilet paper folded 50 times is approximately 17,769,885 miles thick. That is approximately 74 times the distance between the Earth and the moon.*

Watch the following video "[How folding paper can get you to the moon](http://www.youtube.com/watch?v=AmFMJC45f1Q)"

(<http://www.youtube.com/watch?v=AmFMJC45f1Q> )

### AIR Additional Supports

#### Multimedia to enhance comprehension

At times, the use of multimedia can be a valuable way to enrich the instructional experiences that happen in the classroom. Multimedia can be used to reinforce key concepts, to explain ideas in a new way, and to help students further deepen their understandings of main ideas. Teachers could use short snippets or still shots from the video to support ELLs' understanding of the process. Using subtitles with a video clip will provide ELLs with more support. For other videos that may be available, there may be subtitles available in other languages (that students speak) that would also help ELLs with lower levels of proficiency to understand oral English.

To set the stage for the example, have the students preview this activity by first watching the short video, *How Folding Paper Can Get You to the Moon*:

<https://www.youtube.com/watch?v=AmFMJC45f1Q>.

#### Background knowledge for teachers

Some students may belong to communities that find the discussion of items like toilet paper to be uncomfortable and inappropriate. Substitute the term *tissue paper* for toilet paper, and provide an example of a facial tissue.

## Common Core Inc. Exercise 2

### Exercise 2

A rare coin appreciates at a rate of 5.2% a year. If the initial value of the coin is \$500, after how many years will its value cross the \$3,000 mark? Show the formula that will model the value of the coin after  $t$  years.

*The value of the coin will cross the \$3,000 mark between 35 and 36 years;*

$$f(t) = 500(1.052)^t.$$

#### AIR Additional Supports

##### Clarification of key concepts and background knowledge for students

To familiarize ELLs with the concepts and content of this problem, reworking the wording of the problem (without sacrificing rigor or content) is essential.

##### AIR Routine for Teachers

**T:** The verb appreciate mean has several meanings. In mathematics class, it means that something increases in value over time. The amount it appreciates (or increases in value) is called the appreciation.

Suggested rewording: When something is rare or unique, it is usually worth a lot of money. When rare or unique things get older, they are usually worth even more money. For example, last year, a rare coin was worth \$500. Now the same coin is worth \$526. When something is worth more money over time it is called appreciation. The coin appreciated at a rate of 5.2% each year.

- In how many years will the coin be worth more than \$3,000?
- Can you write a formula that models the value after  $t$  years?

## Common Core Inc. Closing

- Consider the sequences  $G(n) = ab^n$ , where  $n$  begins at 0 and  $A(n) = a + bn$ , where  $n$  begins at 0. Assume that  $b > 1$ .
- Which sequence will have the larger 0<sup>th</sup> term? Does it depend on what values are chosen for  $a$  and  $b$ ?

Both sequences will have the same 0<sup>th</sup> term; the term will be  $a$ , regardless of what values are chosen for  $a$  and  $b$ .

#### AIR Additional Supports

##### Teacher modeling and explanation

To assist students, model the closing activity using the table that follows. Have students select a value for  $a$  and  $b$ , where  $b$  is greater than 1. Students should choose different values such that there are multiple examples to draw from during the discussion.

$a =$				
$b =$				
$n$	$G(n) = ab^n$	Think about what happens to each of these sequences when $n = 0$ and $b > 1$ .	$n$	$A(n) = a+bn$
0			0	
1			1	
2			2	

**AIR Additional Supports****Structured opportunities to speak with a partner or small group**

Have students work with their shoulder partner to compare their findings. As a whole group, discuss the big idea (exponential functions grow faster than arithmetic functions).

**Common Core Inc. Exit Ticket**

Chain emails are emails with a message suggesting you will have good luck if you forward the email on to others. Suppose a student started a chain email by sending the message to 3 friends and asking those friends to each send the same email to 3 more friends exactly 1 day after they received it.

- Write an explicit formula for the sequence that models the number of people who will receive the email on the  $n^{\text{th}}$  day. (Let the first day be the day the original email was sent.) Assume everyone who receives the email follows the directions.
- Which day will be the first day that the number of people receiving the email exceeds **100**?

**AIR Additional Supports****Clarification of key concepts and use graphic organizers and foldables**

At times, it may be beneficial to streamline language such that the main ideas (key concepts) are made very clear. This can be done through using present tense, providing shorter sentences with familiar contexts, and ensuring that the visual presentation of the text is clearly organized.

For ELLs, provide the graphic organizer that follows to make sense of the content of this problem. Use modified language that follows to help focus attention on key conceptual ideas.

- Complete the following table.

Day	1	2	3	4	$n$
Number of e-mails sent	3				

- Write a formula that models the number of people who will receive the email on the  $n^{\text{th}}$  day. (Let Day 1 be the day the original e-mail was sent.) Assume everyone who receives the e-mail follows the directions.
- When will more than 100 people receive the chain e-mail? (On which day?)

## Common Core Inc. Exit Ticket Sample Solutions

Chain emails are emails with a message suggesting you will have good luck if you forward the email on to others. Suppose a student started a chain email by sending the message to **3** friends and asking those friends to each send the same email to **3** more friends exactly **1** day after they received it.

- a. Write an explicit formula for the sequence that models the number of people who will receive the email on the  $n^{\text{th}}$  day. (Let the first day be the day the original email was sent.) Assume everyone who receives the email follows the directions.

$$f(n) = 3^n, n \geq 1$$

- b. Which day will be the first day that the number of people receiving the email exceeds **100**?

*On the 5<sup>th</sup> day.*

## Common Core Inc. Problem Set Sample Solutions

1. A bucket is put under a leaking ceiling. The amount of water in the bucket doubles every minute. After 8 minutes, the bucket is full. After how many minutes is the bucket half full?

**7** minutes

2. A three-bedroom house in Burbville was purchased for \$190,000. If housing prices are expected to increase 1.8% annually in that town, write an explicit formula that models the price of the house in  $t$  years. Find the price of the house in 5 years.

$$f(t) = 190,000(1.018)^t, \text{ so } f(5) = 190,000(1.018)^5 = \mathbf{\$207,726.78}.$$

3. A local college has increased the number of graduates by a factor of 1.045 over the previous year for every year since 1999. In 1999, 924 students graduated. What explicit formula models this situation? Approximately how many students will graduate in 2014?

$$f(t) = 924(1.045)^t, \text{ so } f(15) = 924(1.045)^{15} = \mathbf{1,788} \text{ graduates are expected in 2014.}$$

4. The population growth rate of New York City has fluctuated tremendously in the last 200 years, the highest rate estimated at 126.8% in 1900. In 2001, the population of the city was 8,008,288, up 2.1% from 2000. If we assume that the annual population growth rate stayed at 2.1% from the year 2000 onward, in what year would we expect the population of New York City to have exceeded 10 million people? Be sure to include the explicit formula you use to arrive at your answer.

$$f(t) = 8,008,288(1.021)^t. \text{ Based on this formula, we can expect the population of New York City to exceed ten million people in 2012.}$$

5. In 2013, a research company found that smartphone shipments (units sold) were up 32.7% worldwide from 2012, with an expectation for the trend to continue. If 959 million units were sold in 2013, how many smartphones can be expected to be sold in 2018 at the same growth rate? (Include the explicit formula for the sequence that models this growth.) Can this trend continue?

$$f(t) = 959(1.327)^t; f(5) = 959(1.327)^5 = \mathbf{3,946}. \text{ Approximately 3.95 billion units are expected to be sold in 2018. No. There are a finite number of people on Earth, so this trend cannot continue.}$$

6. Two band mates have only 7 days to spread the word about their next performance. Jack thinks they can each pass out 100 fliers a day for 7 days and they will have done a good job in getting the news out. Meg has a different strategy. She tells 10 of her friends about the performance on the first day and asks each of her 10 friends to each tell a friend on the second day and then everyone who has heard about the concert to tell a friend on the third day and so on, for 7 days. Make an assumption that students make sure they are telling someone who has not already been told.

- a. Over the first 7 days, Meg's strategy will reach fewer people than Jack's. Show that this is true.

*Jack's strategy:  $J(t) = 100/\text{day} \times 7 \text{ days} = 700$  people will know about the concert.*

*Meg's strategy:  $M(t) = 10(2)^{t-1}$ ;  $M(7) = 640$  people will know about the concert.*

- b. If they had been given more than 7 days, would there be a day on which Meg's strategy would begin to inform more people than Jack's strategy? If not, explain why not. If so, which day would this occur on?

*On the 8<sup>th</sup> day, Meg's strategy would reach more people than Jack's:*

*$J(8) = 800$ ;  $M(8) = 1280$ .*

- c. Knowing that she has only 7 days, how can Meg alter her strategy to reach more people than Jack does?

*She can ask her ten initial friends to tell two people each and let them tell two other people on the next day, and so on.*

7. On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

- a. When will the lake be covered half way?

*June 29*

- b. On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?

*On June 26, the lake will only be 6.25% covered. To the casual observer, it will be hard to imagine such a jump between this small percent of coverage to 100% coverage in merely 4 more days.*

- c. On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

*It only takes care of the problem for a week:*

*June 29–1%*

*June 30–2%*

*July 1–4%*

*July 2–8%*

*July 3–16%*

*July 4–32%*

*July 5–64%*

By July 6, the lake will be completely covered with algae.

- d. Write an explicit formula for the sequence that models the percentage of the surface area of the lake that is covered in algae,  $a$ , given the time in days,  $t$ , that has passed since the algae was introduced into the lake.

$$f(t) = a(2^{(t-1)})$$

8. Mrs. Davis is making a poster of math formulas for her students. She takes the 8.5 in.  $\times$  11 in. paper she printed the formulas on to the photocopy machine and enlarges the image so that the length and the width are both 150% of the original. She enlarges the image a total of 3 times before she is satisfied with the size of the poster. Write an explicit formula for the sequence that models the area of the poster,  $A$ , after  $n$  enlargements. What is the area of the final image compared to the area of the original, expressed as a percent increase and rounded to the nearest percent?

*The area of the original piece of paper is  $93.5 \text{ in}^2$ . Increasing the length and width by a factor of  $1.5$  increases the area by a factor of  $2.25$ . Thus,  $A(n) = 93.5(2.25)^n$ . The area after 3 iterations is approximated by  $93.5(11.39)$  for a result of  $1065 \text{ in}^2$ . The percent increase was  $1039\%$ .*

## AIR Additional Supports

### Homework scaffolds

Students in mathematics are increasingly tasked with communicating their mathematical thinking in ways that move beyond simple articulation of steps taken to solve problems. In the case of ELLs, teachers will need to consistently monitor and support students' academic language levels and provide opportunities for students to share their thinking in meaningful ways, which may include through the use of demonstrations (perhaps recorded via screencast or video), sketches, or word banks and sentence frames to express ideas. Collaboration between mathematics teachers and ESOL or bilingual teachers for specific information about supporting individual students can be beneficial. ELLs literate in their first language should be given opportunities to read and write in that language when possible as a way to deepen content understandings and while perhaps also engaging the families of students in this work. The homework assignment for this lesson has been rewritten with the same rigor but with a scaffolded language load to help clarify the key concepts of the lesson.

### Graphic organizers and foldables

For each item in the problem set, have students create a table to organize their solutions. For example, in problem 2, a two-column table should be created to indicate the year (left column) and the value (right column). In problem 3, a two-column table should be created to indicate the year (left column) and the number of students (right column). This approach can be used in each of the problems to organize the data.

### Concrete and visual models

Because many of the terms used in the sample set are not commonly known to ELLs at the entering and emerging levels, create a quick sketch or search for an image to represent the context.

### Clarification of key concepts

Each problem can have the language load scaffolded to make the concepts more accessible, either by framing the contexts in the present tense or by rearranging the conditional statements, or by choosing more high-frequency vocabulary, all without sacrificing the intended rigor.

**Alternate Problem Set**

The italicized text that follows can be used in place of the text for the problems as indicated in the original lesson. Problem 1 would be made more accessible through the use of an image of a bucket, because this word (bucket) is not typically among those first learned by ELLs.

**Problem 1:**

Include an image of a bucket.

**Problem 2**

*Natalia buys a house for \$190,000. Each year, the value of the house will increase by 1.8%. Write a formula to model the price of the house in  $t$  years. Find the price of the house in 5 years.*

**Problem 3**

*In the year 1999, 924 students graduated from a university. Every year, the university increases the number of graduates by a factor of **1.045**. What formula models this situation? Approximately how many students will graduate in 2014?*

**Problem 4**

*In the year 2001, the population of New York City was **8,008,288**. This number is **2.1%** greater than the population of New York City in the year 2000. In what year would the population of New York City be more than 10 million people if the growth rate stays at 2.1%? Write a formula to show your answer.*

**Problem 5**

*In the year 2013, 959 million smartphones were sold. This number of was an increase of **32.7%** from the number sold in the year 2012. How many smartphones will be sold in 2018 if the growth rate is 32.7% every year? Do you think smartphones sales will continue to grow at this rate? Why or why not? Explain your thinking.*

**Problem 6**

*Jack and Meg are having a concert. They want lots of people to attend. The concert is in 7 days. Jack and Meg have different ideas (strategies) to tell people about the concert:*

*Jack passes out **100** fliers a day for **7** days.*

*On the first day, Meg tells **10** of her friends about the concert. On the second day, each of Meg's friends tells 10 more people about the concert. On the third day, each person tells another friend about the concert. This pattern continues for **7** days.*

- a. How many people know about the concert using Jack's strategy? How many people know about the concert using Meg's strategy? Write a formula that models each strategy.*
- b. On which day does Meg's strategy reach more people than Jack's strategy?*
- c. How can Meg change her strategy to reach more people than Jack does over 7 days?*



**Problem 7**

Algae starts growing on the surface of a lake on June 1. Each day, the amount of algae on the surface of the lake doubles in size. If the algae continues to grow, the lake will be totally covered on June 30.

- When will the lake be covered half way?
- A man walks by the lake every day. On June 26, he tells his friends that the lake will be completely covered with algae soon. The friend does not believe him. Why not?
- On June 29, people come to clean up the algae on the lake. When they are done cleaning, only 1% of the surface is covered with algae. Does this solve the problem of the algae in the lake? Use the table to explain why or why not.

Date	June 29	June 30	July 1	July 2	July 3	July 4	July 5	July 6
% covered	1	2						

- Write a formula that models the percentage of the surface area of the lake that is covered in algae,  $a$ , given the time in days,  $t$ , that have passed since the algae started growing.

**Problem 8**

Mrs. Davis is making a poster of mathematics formulas for her students. She prints the formulas on a standard piece of paper that measures 8.5 inches by 11 inches. Mrs. Davis enlarges (increases the size of) the poster so that the length and width are both 150% of the original. Mrs. Davis still thinks the poster is too small. She enlarges the poster two more times by a factor of 150%.

- By what factor is the area of the poster enlarged each time?
- Write a formula that models the area of the poster,  $A$ , after  $n$  enlargements.
- What is the percentage of increase of the poster's area from its original size?

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